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Towards covariant quantization of the supermembrane

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Abstract: By replacing ten-dimensional pure spinors with eleven-dimensional pure spinors, the formalism recently developed for covariantly quantizing the $d = 10$ superparticle and superstring is extended to the $d = 11$ superparticle and supermembrane. In this formalism, kappa symmetry is replaced by a BRST-like invariance using the nilpotent operator $Q = \oint \lambda^\alpha d_\alpha$ where d_α is the worldvolume variable corresponding to the $d = 11$ spacetime supersymmetric derivative and λ^α is an SO(10,1) pure spinor variable satisfying $\lambda\Gamma^c\lambda = 0$ for $c = 1$ to 11. Super-Poincaré covariant unintegrated and integrated supermembrane vertex operators are explicitly constructed which are in the cohomology of Q. After double-dimensional reduction of the eleventh dimension, these vertex operators are related to type-IIA superstring vertex operators where $Q = Q_L + Q_R$ is the sum of the left and right-moving type-IIA BRST operators and the eleventh component of the pure spinor constraint, $\lambda\Gamma^{11}\lambda = 0$, replaces the $b_L^0 - b_R^0$ constraint of the closed superstring. A conjecture is made for the computation of M-theory scattering amplitudes using these supermembrane vertex operators.

Keywords: BRST Quantization, M-Theory, Superstrings and Heterotic Strings.

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1. Introduction

There is substantial evidence that $d = 10$ superstring theory contains nonperturbative symmetries coming from an underlying $d = 11$ theory which has been named M-theory [1, 2]. It is believed that M-theory properties are related to the supermembrane [3, 4, 5, 6], however, problems with covariantly quantizing the supermembrane have made it difficult to study these properties. Although there exist light-cone methods such as M(atrix) theory [7, 8] for studying the supermembrane, the lack of spacetime gauge and Lorentz symmetries makes these light-cone methods clumsy and limits their use to special backgrounds. Nevertheless, certain properties of M-theory have been successfully studied using light-cone supermembrane vertex operators [9], and it should now be possible to covariantize these light-cone methods using the results of this paper.

There are two essential problems with covariant quantization of the supermembrane which, naively, appear to be unrelated. The first problem, which is also present in the bosonic membrane, is the complicated non-quadratic nature of the supermembrane hamiltonian and the resulting difficulties in constructing the physical spectrum. The second problem, which is also present in the Green-Schwarz (GS) superstring [10], is the kappa symmetry [11] of the supermembrane action which implies fermionic second-class constraints that are difficult to covariantly separate out from the first-class constraints.

Recently, a new formalism [12][13] was developed for quantizing the superstring which preserves manifest SO(9,1) super-Poincaré covariance but does not suffer from the problems of the GS formalism. This formalism uses a new version of the superstring action which includes bosonic pure spinor ghost variables λ^μ satisfying $\lambda\gamma^m\lambda = 0$ for $m = 1$ to 10.[1] In this pure spinor formalism for the superstring, kappa symmetry is replaced by a BRST-like invariance using the nilpotent operator $Q = \oint \lambda^\mu d_\mu$ where d_μ is the worldsheet variable for the $d = 10$ spacetime supersymmetric derivative. Physical vertex operators are defined as states in the cohomology of Q and, since the worldsheet action is quadratic, manifestly super-Poincaré covariant scattering amplitudes [14] can be computed using the free-field OPE's of the worldsheet variables.

Since the standard supermembrane action [4] reduces to the GS version of the type-IIA superstring action [10] after double-dimensional reduction of the eleventh dimension [5], it is natural to look for an alternative version of the supermembrane action which reduces instead to the pure spinor version of the type-IIA superstring action. Such a generalization is reasonable given the results of [15] where $d = 10$ twistor-like methods for the superstring were generalized to $d = 11$ twistor-like methods for the supermembrane. Furthermore, it was shown by Howe that just as the super-Yang-Mills equations of motion can be understood as $d = 10$ pure spinor integrability conditions [16], the $d = 11$ supergravity equations of motion can be understood as $d = 11$ pure spinor integrability conditions [17].

As will be shown in this paper, it is indeed possible to construct a supermembrane action which reduces after double-dimensional reduction to the pure spinor version of the

[1] To simplify $d = 11$ language, the time coordinate will be called x^{10} instead of x^0. The indices $m, n, p, \ldots$ and $\mu, \nu, \rho, \ldots$ will label $d = 10$ vectors and spinors, and the indices $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ will label $d = 11$ vectors and spinors.

type-IIA superstring action. In this pure spinor version of the supermembrane action, kappa symmetry is replaced by a BRST-like invariance using the nilpotent operator $Q = \oint \lambda^\alpha d_\alpha$ where d_α is now the worldvolume variable for the $d = 11$ spacetime supersymmetric derivative and λ^α is an $\mathrm{SO}(10,1)$ pure spinor ghost variable satisfying $\lambda\Gamma^c\lambda = 0$ for $c = 1$ to 11.[2] After double-dimensional reduction of the eleventh dimension, d_α splits into the left and right-moving type-IIA worldsheet variables $d_{L\mu}$ and $d_{R\hat{\mu}}$, λ^α splits into the left and right-moving type-IIA pure spinor variables λ^μ_L and $\lambda^{\hat{\mu}}_R$, and Q reduces to the sum of the left and right-moving type-IIA BRST operators, $Q = Q_L + Q_R$ where $Q_L = \oint \lambda^\mu_L d_{L\mu}$ and $Q_R = \oint \lambda^{\hat{\mu}}_R d_{R\hat{\mu}}$. Furthermore, the eleventh component of the pure spinor constraint, $\lambda\Gamma^{11}\lambda = 0$, replaces the $b^0_L - b^0_R$ constraint[3] which is necessary for defining BRST cohomology in closed string theory [18].

Since kappa symmetry is replaced by a BRST-like invariance, this formalism does not suffer from quantization problems associated with second-class constraints. Furthermore, since physical states will be defined as states in the cohomology of Q, the complicated nature of the supermembrane hamiltonian does not directly enter into the computation of the physical spectrum. Note that it was proven for the superstring that states in the cohomology of Q are annihilated by the hamiltonian [19], and one expects that this will also be true for the supermembrane. It might seem surprising that quantization of the supermembrane is simpler than quantization of the bosonic membrane, but this situation often occurs in supersymmetric systems where second-order differential equations can be replaced by first-order differential equations.

Before discussing the supermembrane, it will be useful to first covariantly quantize the $d = 11$ superparticle which describes the zero modes of the supermembrane. To make this paper self-contained, covariant quantization of the $N = 1$ and $N = 2$ $d = 10$ superparticles and type-II superstring will also be reviewed here.

As reviewed in section 2, covariant quantization [13] of the $N = 1$ $d = 10$ superparticle allows a BRST description of super-Maxwell theory where the nilpotent BRST operator is $Q = \lambda^\mu d_\mu$, d_μ is the $N = 1$ $d = 10$ supersymmetric derivative, and λ^μ is a $d = 10$ pure spinor ghost variable satisfying $\lambda\gamma^m\lambda = 0$ for $m = 1$ to 10. Using a suitably defined norm $\langle\ \ \rangle$ of ghost number three, the super-Maxwell action can be constructed as $\int d^{10}x \langle \Psi Q \Psi \rangle$ where $\Psi(\lambda, x, \theta)$ is a quantum-mechanical wavefunction depending on the $d = 10$ pure spinor and superspace variables. At ghost number one, $\Psi = \lambda^\mu A_\mu(x,\theta)$ where $A_\mu(x,\theta)$ describes the super-Maxwell fields, and at other ghost numbers, Ψ describes the super-Maxwell ghosts, antifields and antighosts.

Furthermore, by coupling the pure spinor version of the $N = 1$ $d = 10$ superparticle action to a super-Maxwell background, one obtains the integrated version of the open superstring massless vertex operator. By evaluating correlation functions of these integrated vertex operators with the unintegrated massless vertex operators $\Psi = \lambda^\mu A_\mu$, one can com-

[2]The definition of $d = 11$ pure spinors used here differs from that of Howe in [17] where $d = 11$ pure spinors were required to satisfy both $\lambda\Gamma^c\lambda = 0$ and $\lambda\Gamma^{cd}\lambda = 0$. Howe's definition of a $d = 11$ pure spinor is more restrictive than the definition used here and does not appear to be appropriate for superparticle and supermembrane quantization.

[3]I thank Barton Zwiebach for stressing the importance of the $b^0_L - b^0_R$ constraint.

pute $N = 1$ $d = 10$ supersymmetric Born-Infeld amplitudes in a manifestly super-Poincaré covariant manner. The normalization for the worldsheet zero modes in these correlation functions is defined by the ghost number three norm used in the super-Maxwell action.

In section 3, the $N = 2$ $d = 10$ superparticle is covariantly quantized using the BRST operators $Q_L = \lambda_L^\mu d_{L\mu}$ and $Q_R = \lambda_R^{\hat\mu} d_{R\hat\mu}$ where $d_{L\mu}$ and $d_{R\hat\mu}$ are the $N = 2$ $d = 10$ supersymmetric derivatives and λ_L^μ and $\lambda_R^{\hat\mu}$ are independent pure spinors satisfying $\lambda_L \gamma^m \lambda_L = \lambda_R \gamma^m \lambda_R = 0$. At non-zero momentum, the physical spectrum corresponds to linearized type-II supergravity, however, at zero momentum, there are type-II supergravity states that are missing from the $N = 2$ $d = 10$ superparticle spectrum. This fact is related to the absence of a $b_L^0 - b_R^0$ constraint, which is known from closed string field theory [18] to be necessary for obtaining the correct physical spectrum at zero momentum. The absence of the $b_L^0 - b_R^0$ constraint in the $N = 2$ $d = 10$ superparticle also prevents the construction of a $\int d^{10}x \langle \Psi Q \Psi \rangle$ action for linearized type-II supergravity, which is not surprising for the type-IIB superparticle because of the self-dual five-form field strength in the spectrum.

In section 4, a pure spinor version of the $d = 11$ superparticle action is constructed. In this action, kappa symmetry is replaced by a BRST-like invariance generated by the nilpotent operator $Q = \lambda^\alpha d_\alpha$ where d_α is the $d = 11$ supersymmetric derivative and λ^α is a $d = 11$ pure spinor satisfying $\lambda \Gamma^c \lambda = 0$ for $c = 1$ to 11. Using the results of the appendix where the zero momentum cohomology of Q is explicitly computed, it is argued that the complete cohomology of Q describes linearized $d = 11$ supergravity [20]. As in the super-Maxwell action constructed using the $N = 1$ $d = 10$ superparticle, the linearized $d = 11$ supergravity action can be constructed as $\int d^{11}x \langle \Psi Q \Psi \rangle$ where $\Psi(\lambda, x, \theta)$ is a quantum-mechanical wavefunction depending on the $d = 11$ pure spinor and superspace variables, and $\langle\ \ \rangle$ is a suitably defined norm of ghost number seven. At ghost number three, $\Psi(\lambda, x, \theta) = \lambda^\alpha \lambda^\beta \lambda^\gamma A_{\alpha\beta\gamma}(x, \theta)$ where $A_{\alpha\beta\gamma}(x, \theta)$ describes the linearized $d = 11$ supergravity fields, and at other ghost numbers, Ψ describes the linearized $d = 11$ supergravity ghosts, antifields and antighosts. The fact that $d = 11$ supergravity fields carry ghost number three is explained by their coupling to the three-dimensional worldvolume of the supermembrane, while the ghost number one super-Maxwell fields and ghost number two type-II supergravity fields couple respectively to the one-dimensional superparticle worldline and two-dimensional superstring worldsheet.

When $P_{11} = 0$, the $d = 11$ superparticle BRST operator reduces to $Q = Q_L + Q_R$ where Q_L and Q_R are the $N = 2$ $d = 10$ superparticle BRST operators, and the physical spectrum is linearized type-IIA supergravity without the zero momentum problems that were encountered using the $N = 2$ $d = 10$ superparticle. This is possible since the $b_L^0 - b_R^0$ constraint is imposed in the $d = 11$ superparticle by the eleventh component of the pure spinor constraint, $\lambda \Gamma^{11} \lambda = 0$, which is not present in the $N = 2$ $d = 10$ superparticle.

In section 5, covariant quantization of the $N = 2$ $d = 10$ superparticle is generalized to the type-II superstring by extending the pure spinor and $N = 2$ $d = 10$ superspace variables to worldsheet fields. After reviewing the pure spinor version of the closed superstring action in a flat background, a BRST-invariant action is constructed in a curved type-II supergravity background where the left and right-moving BRST operators, $Q_L = \oint \lambda_L^\mu d_{L\mu}$ and $Q_R = \oint \lambda_R^{\hat\mu} d_{R\hat\mu}$, are conserved and nilpotent when the curved background is on-shell [21].

The integrated form of the closed superstring massless vertex operator is the linearized contribution of the curved background to the superstring action, and the unintegrated form of the massless vertex operator is the $N = 2$ superparticle wavefunction $\Psi(\lambda_L, \lambda_R, x, \theta_L, \theta_R)$. Using the left-right product of the zero mode normalization of the $N = 1$ $d = 10$ superparticle, one can compute type-II superstring massless tree amplitudes in a manifestly super-Poincaré covariant manner by evaluating the correlation function of these integrated and unintegrated massless vertex operators on a string worldsheet. In a flat background, the pure spinor version of the superstring action is quadratic, so these correlation functions can easily be evaluated using the free field OPE's of the worldsheet variables.

Finally, in section 6, a pure spinor version of the $d = 11$ supermembrane action is constructed where kappa symmetry is replaced by a BRST-like invariance generated by $Q = \oint \lambda^\alpha d_\alpha$ and λ^α and d_α are now worldvolume variables. This supermembrane action reduces to the pure spinor version of the $d = 11$ superparticle action when the membrane tension becomes infinite, and reduces to the pure spinor version of the type-IIA superstring action when the eleventh dimension is compactified on an infinitesimally small circle keeping the string tension constant. The supermembrane action is then generalized to a curved $d = 11$ supergravity background where the BRST operator $Q = \oint \lambda^\alpha d_\alpha$ is nilpotent and conserved when the background is on-shell.

As in the superstring, the integrated form of the massless supermembrane vertex operator is the linearized contribution of the curved background to the action, and the unintegrated form of the massless supermembrane vertex operator is the $d = 11$ superparticle wavefunction $\Psi(\lambda, x, \theta)$. Using the same zero mode normalization as in the $d = 11$ superparticle, one can formally define supermembrane scattering amplitudes as correlation function of these integrated and unintegrated vertex operators on a membrane worldvolume. Although the supermembrane action is not quadratic in a flat background, it might be possible to compute these correlation functions using a perturbative expansion in the inverse of the membrane tension. Note that unlike superstring scattering amplitudes, one does not expect a genus expansion for supermembrane scattering amplitudes since there is no coupling of worldvolume curvature to a spacetime field in the supermembrane action.

In section 7, a conjecture is made that these supermembrane scattering amplitudes are M-theory scattering amplitudes which, after compactification of the eleventh dimension on a circle whose radius depends on the string coupling constant, reproduce type-IIA superstring scattering amplitudes. Since the perturbative expansion in the membrane tension preserves manifest $d = 11$ super-Poincaré covariance, these scattering amplitudes would contain non-perturbative information about the type-IIA superstring which might be useful for studying M-theory.

2. Covariant quantization of the $N = 1$ $d = 10$ superparticle

Since the $d = 11$ superparticle has a simpler action than the $d = 11$ supermembrane, it will be useful to explain how to covariantly quantize the $d = 11$ superparticle before discussing the supermembrane. The quantization method is similar to the method used in [13] for quantizing the $N = 1$ $d = 10$ superparticle, which will be reviewed first.

2.1 Standard description of the $N = 1$ $d = 10$ superparticle

The standard action for the $N = 1$ $d = 10$ superparticle is [22]

$$S = \int d\tau (P_m \Pi^m + e P^m P_m) \tag{2.1}$$

where

$$\Pi^m = \dot{x}^m + \frac{i}{2}\theta^\mu \gamma^m_{\mu\nu} \dot{\theta}^\nu \,, \tag{2.2}$$

$m = 1$ to 10 is the SO(9,1) vector index with x^{10} as the time coordinate, $\mu = 1$ to 16 is the SO(9,1) Majorana-Weyl spinor index, P_m is the canonical momentum for x^m, and e is the Lagrange multiplier which enforces the mass-shell condition. The gamma matrices $\gamma^m_{\mu\nu}$ and $\gamma^{m\mu\nu}$ are 16×16 symmetric matrices which satisfy $\gamma^{(m}_{\mu\nu}\gamma^{n)\ \nu\rho} = 2\eta^{mn}\delta^\rho_\mu$. Upper spinor indices will denote Weyl $d = 10$ spinors whereas lower spinor indices will denote anti-Weyl $d = 10$ spinors. In terms of the standard 32×32 $d = 10$ Γ-matrices satisfying $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}$, $\gamma^m_{\mu\nu}$ and $\gamma^{m\mu\nu}$ are the off-diagonal blocks of Γ^m in the Weyl representation. Note that any $d = 10$ antisymmetric bispinor $f^{[\mu\nu]}$ can be written in terms of a three-form as $f^{[\mu\nu]} = (\gamma_{mnp})^{\mu\nu} f^{mnp}$, and any $d = 10$ symmetric bispinor $g^{(\mu\nu)}$ can be decomposed into a one-form and five-form as $g^{(\mu\nu)} = \gamma^{\mu\nu}_m g^m + (\gamma_{mnpqr})^{\mu\nu} g^{mnpqr}$. Furthermore, the $d = 10$ gamma matrices satisfy the identity $\eta_{mn}\gamma^m_{(\mu\nu}\gamma^n_{\rho)\sigma} = 0$.

The action of (2.1) is invariant under the global $N = 1$ $d = 10$ spacetime-supersymmetry transformations

$$\delta\theta^\mu = \epsilon^\mu \,, \qquad \delta x^m = \frac{i}{2}\theta\gamma^m\epsilon \,, \qquad \delta P_m = \delta e = 0 \,, \tag{2.3}$$

and under the local kappa transformations [11]

$$\delta\theta^\mu = P^m(\gamma_m\kappa)^\mu \,, \qquad \delta x^m = -\frac{i}{2}\theta\gamma^m\delta\theta \,, \qquad \delta P_m = 0 \,, \qquad \delta e = i\dot{\theta}^\nu\kappa_\nu \,. \tag{2.4}$$

The canonical momentum to θ^μ, which will be called p_μ, satisfies

$$p_\mu = \frac{\partial L}{\partial \dot{\theta}^\mu} = \frac{i}{2}P^m(\gamma_m\theta)_\mu \,,$$

so canonical quantization requires that physical states are annihilated by the sixteen fermionic Dirac constraints defined by

$$d_\mu = p_\mu - \frac{i}{2}P_m(\gamma^m\theta)_\mu \,. \tag{2.5}$$

Since $\{p_\mu, \theta^\nu\} = -i\delta^\nu_\mu$, these constraints satisfy the Poisson brackets

$$\{d_\mu, d_\nu\} = -P_m\gamma^m_{\mu\nu} \,, \tag{2.6}$$

and since $P^m P_m = 0$ is also a constraint, eight of the sixteen Dirac constraints are first-class and eight are second-class. One can easily check that the eight first-class Dirac constraints generate the kappa transformations of (2.4), however, there is no simple way to covariantly separate out the second-class constraints.

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Although one cannot covariantly quantize the action of (2.1), one can classically couple the superparticle to a super-Maxwell background using the action

$$\widehat{S}=\int d\tau\left[P_m\Pi^m+eP^mP_m+q\left(\dot{\theta}^\mu A_\mu(x,\theta)+\Pi^m A_m(x,\theta)\right)\right] \tag{2.7}$$

where A_μ and A_m are the spinor and vector super-Maxwell gauge superfields and q is the charge of the superparticle. The action of (2.7) is invariant under spacetime supersymmetry and under the background gauge transformations $\delta A_\mu = D_\mu\Lambda$ and $\delta A_m = \partial_m\Lambda$ where $D_\mu = \frac{\partial}{\partial\theta^\mu}+\frac{i}{2}(\gamma^m\theta)_\mu\partial_m$. And if the kappa transformations of (2.4) are modified to

$$\delta\theta^\mu = P^m(\gamma_m\kappa)^\mu\,,\quad \delta x^m=-\frac{i}{2}\theta\gamma^m\delta\theta\,,\quad \delta P_m=-q\delta\theta\gamma^m W\,,\quad \delta e = i(\dot{\theta}^\nu+2ieqW^\nu)\kappa_\nu \tag{2.8}$$

where $W^\mu = \frac{1}{10}\gamma^{m\mu\nu}(D_\nu A_m - \partial_m A_\nu)$ is the super-Maxwell spinor field strength superfield [23], the action of (2.7) is invariant under (2.8) when A_μ and A_m satisfy the super-Maxwell equations of motion $D_\mu A_\nu + D_\nu A_\mu = i\gamma^m_{\mu\nu}A_m$.

2.2 Pure spinor description of the $N=1$ $d=10$ superparticle

Instead of using the standard superparticle action of (2.1), the pure spinor formalism for the $N=1$ $d=10$ superparticle uses the quadratic action [13]

$$S_{\text{pure}}=\int d\tau\left(P_m\dot{x}^m+p_\mu\dot{\theta}^\mu+w_\mu\dot{\lambda}^\mu-\frac{1}{2}P^mP_m\right) \tag{2.9}$$

where p_μ is now an independent variable [24], λ^μ is a pure spinor ghost variable satisfying

$$\lambda\gamma^m\lambda=0 \qquad \text{for } m=1 \text{ to } 10\,, \tag{2.10}$$

and w_μ is the canonical momentum to λ^μ which is defined up to the gauge transformation

$$\delta w_\mu = (\gamma^m\lambda)_\mu\Lambda_m \tag{2.11}$$

for arbitrary gauge parameters Λ_m. One can easily show using a U(5) decomposition of a Wick-rotated SO(10) spinor that the constraint of (2.10) and the gauge transformation of w_μ imply that λ^μ and w_μ each contain eleven independent components.[4] The action of (2.9) can be written in manifestly spacetime supersymmetric notation as

$$S_{\text{pure}}=\int d\tau\left(P_m\Pi^m+d_\mu\dot{\theta}^\mu+w_\mu\dot{\lambda}^\mu-\frac{1}{2}P^mP_m\right) \tag{2.12}$$

where Π^m and d_μ are defined as in (2.2) and (2.5). Note that d_μ is defined to be invariant under spacetime supersymmetry, so p_μ should be defined to transform as $\delta p_\mu = \frac{i}{2}P_m(\gamma^m\epsilon)_\mu$ under (2.3).

To obtain the correct physical spectrum, the action of (2.9) needs to be supplemented with the constraint that physical states are in the cohomology of the BRST-like operator

$$Q=\lambda^\mu d_\mu\,. \tag{2.13}$$

[4]Although the eleven independent components of λ^μ must be complex in order to satisfy (2.10), their complex conjugates $\overline{\lambda^\mu}$ never appear in the pure spinor formalism and can therefore be ignored.

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Note that $Q^2 = 0$ using (2.10) and (2.6), and carries ghost-number $+1$ if λ^μ and w_μ are defined to carry ghost-number $+1$ and -1 respectively. Although it is not yet understood how to obtain Q from gauge-fixing a reparameterization invariant action, it is straightforward to covariantly quantize the superparticle using this BRST operator and check that one obtains the correct spectrum.

Unlike the usual particle action where the mass-shell condition comes from the reparameterization constraint $P_m P^m = 0$, the mass-shell condition in the pure spinor formalism is implied indirectly by the $Q = \lambda^\mu d_\mu$ constraint. Furthermore, the gauge invariances generated by Q replace the kappa transformations of (2.4) which are not a symmetry of (2.9). Although light-cone gauge fixing is more subtle in the pure spinor formalism than in the usual formalism, one can check that the correct counting of light-cone variables can be obtained by using the pure spinor ghost variables to cancel the non-physical matter variables. The 22 independent bosonic ghost variables of λ^μ and w_μ cancel 22 of the 32 fermionic variables of θ^μ and p_μ, leaving ten fermionic variables. Two of these ten fermionic variables act as the missing (b, c) reparameterization ghosts and cancel the longtitudinal components of x^m and P_m. The remaining eight fermionic variables are the physical light-cone fermionic variables.

2.3 BRST description of super-Maxwell theory

Using the BRST quantization method, the cohomology of the BRST operator Q at a fixed ghost number should reproduce the physical fields in the spectrum. Furthermore, the structure of BRST transformations implies that if the ghost number of physical fields is defined to be G, the states at ghost number less than G describe spacetime ghosts, the states at ghost number $G+1$ describe spacetime antifields, and the states at ghost number greater than $G+1$ describe spacetime antighosts. This structure comes from the fact that the BRST transformation of a field is its gauge transformation using a ghost as the gauge parameter, the BRST transformation of an antifield is the equation of motion of the corresponding field, and the BRST transformation of an antighost is the gauge-fixing condition acting on the antifield. As will now be reviewed, the cohomology of the BRST operator $Q = \lambda^\mu d_\mu$ for the $N = 1$ $d = 10$ superparticle correctly reproduces these states for $N = 1$ $d = 10$ super-Maxwell theory where the ghost number of physical fields is defined to be $G = 1$.

At ghost-number one, the states in the $N = 1$ $d = 10$ superparticle Hilbert space are described by the wavefunction

$$\Psi(\lambda, x, \theta) = \lambda^\mu A_\mu(x, \theta) \tag{2.14}$$

where $A_\mu(x, \theta)$ is a spinor superfield. Since $\lambda^\mu \lambda^\nu$ is proportional to $(\gamma_{mnpqr})^{\mu\nu} \lambda \gamma^{mnpqr} \lambda$,

$$Q\Psi = \lambda^\mu \lambda^\nu D_\nu A_\mu = 0 \tag{2.15}$$

implies that

$$(\gamma_{mnpqr})^{\mu\nu} D_\mu A_\nu = 0 \tag{2.16}$$

where $D_\mu = \frac{\partial}{\partial\theta^\mu} + \frac{i}{2}(\gamma^m\theta)_\mu\partial_m$ is the $N=1$ $d=10$ supersymmetric derivative and $mnpqr$ is any five-form direction. And

$$\delta\Psi = Q\Lambda = \lambda^\mu D_\mu\Lambda \tag{2.17}$$

implies the gauge transformation

$$\delta A_\mu(x,\theta) = D_\mu\Lambda(x,\theta)\,. \tag{2.18}$$

Eqs. (2.16) and (2.18) are the $N=1$ $d=10$ super-Maxwell equations of motion and gauge invariances written in terms of the spinor gauge superfield $A_\mu(x,\theta)$. To see this, one can expand $A_\mu(x,\theta)$ in components as

$$A_\mu(x,\theta) = f_\mu(x) + f_{\mu\nu}(x)\theta^\nu + f_{\mu\nu\rho}(x)\theta^\nu\theta^\rho + \cdots\,. \tag{2.19}$$

Using the gauge invariance and equations of motion of (2.17) and (2.16), one can set

$$f_\mu(x) = 0\,, \qquad f_{\mu\nu}(x) = \gamma^m_{\mu\nu}a_m(x)\,, \qquad f_{\mu\nu\rho}(x) = \eta_{mn}\gamma^m_{\mu[\nu}\gamma^n_{\rho]\sigma}\chi^\sigma(x)\,, \tag{2.20}$$

and all higher components of $A_\mu(x,\theta)$ to be proportional to $a_m(x)$ and $\chi^\rho(x)$, where $a_m(x)$ and $\chi^\rho(x)$ are the photon and photino satisfying the equations of motion and gauge invariances

$$\partial^m\partial_{[m}a_{n]} = \partial^m(\gamma_m\chi)_\mu = 0\,, \qquad \delta a_m = \partial_m\omega\,. \tag{2.21}$$

So the cohomology of $\lambda^\mu d_\mu$ at ghost number one reproduces the physical super-Maxwell fields. To check that the cohomology at other ghost numbers correctly reproduces the super-Maxwell ghosts, antifields, and antighosts, it is convenient to first compute the cohomology at zero momentum. As shown in the appendix of [13], the cohomology of $Q = \lambda^\mu d_\mu$ at zero momentum is equivalent to the cohomology of

$$\begin{aligned}\widetilde{Q} = {}& \widetilde{\lambda}^\mu p_\mu + (\widetilde{\lambda}\gamma^m\widetilde{\lambda})b_{(-1)m} + c^m_{(1)}(\widetilde{\lambda}\gamma_m)_\mu u^\mu_{(-1)} + (\widetilde{\lambda}\gamma_m\widetilde{\lambda})(b^\mu_{(-2)}\gamma^m_{\mu\nu}v^\nu_{(1)}) - \\ & - 2(b_{(-2)\mu}\widetilde{\lambda}^\mu)(v_{(1)\nu}\widetilde{\lambda}^\nu) + c^\mu_{(2)}(\gamma_m\widetilde{\lambda})_\mu u^m_{(-2)} + (\widetilde{\lambda}\gamma^m\widetilde{\lambda})v_{(2)m}b_{(-3)}\,,\end{aligned} \tag{2.22}$$

where $\widetilde{\lambda}^\mu$ is an unconstrained spinor and $[b_{(-n)}, c_{(n)}]$ and $[u_{(-n)}, v_{(n)}]$ are new fermionic and bosonic pairs of conjugate variables of ghost number $[-n,n]$ which cancel the effect of removing the pure spinor constraint on λ^μ.[5] The term $b_{(-1)m}(\widetilde{\lambda}\gamma^m\widetilde{\lambda})$ replaces the pure spinor constraint, and the other terms in $\widetilde{Q}$ are needed to eliminate the extra gauge invariances implied by this constraint. For example, since $b_{(-1)m}(\widetilde{\lambda}\gamma^m\widetilde{\lambda})$ is invariant under $\delta b_{(-1)m} = \widetilde{\lambda}\gamma^m f$, one needs to include the term $c^m_{(1)}(\widetilde{\lambda}\gamma_m)_\mu u^\mu_{(-1)}$. And since this term is invariant under $\delta u^\mu_{(-1)} = (\widetilde{\lambda}\gamma_m\widetilde{\lambda})(\gamma^m f)^\mu - 2\widetilde{\lambda}^\mu(\widetilde{\lambda}^\nu f_\nu)$, one needs to include the term $(\widetilde{\lambda}\gamma_m\widetilde{\lambda})(b^\mu_{(-2)}\gamma^m_{\mu\nu}v^\nu_{(1)}) - 2(b_{(-2)\mu}\widetilde{\lambda}^\mu)(v_{(1)\nu}\widetilde{\lambda}^\nu)$.

[5]This approach of adding new variables and removing the pure spinor constraint was recently used to quantize the superstring in [25]. However, at non-zero momentum, this approach leads to complications which makes it usefulness unclear. For example, one needs to include a term $c^m_{(1)}P_m$ in (2.22) which naively puts a constraint on the momentum P_m.

Since $\widetilde{\lambda}^\mu$ is unconstrained, it is easy to compute the zero-momentum cohomology of $\widetilde{Q}$ at arbitrary ghost number. One finds that the states in the cohomology are in one-to-one correspondence with the variables $[1, c^m_{(1)}, v_{(1)\mu}, c^\mu_{(2)}, v_{(2)m}, c_{(3)}]$. So there is a scalar spacetime ghost at ghost number zero, a vector and spinor field at ghost number one, a spinor and vector antifield at ghost number two, and a scalar spacetime antighost at ghost number three. This reproduces the desired BRST structure of super-Maxwell theory since the only gauge field is the photon which implies a single scalar ghost. Using the map between Q and $\widetilde{Q}$, one finds that the corresponding states in the zero-momentum cohomology of Q with constrained λ^μ are given by

$$\begin{aligned}\Psi(\lambda,\theta) = {}& \omega + (\lambda\gamma^m\theta)a_m + (\lambda\gamma^m\theta)(\theta\gamma_m\chi) + (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mn}\chi^*) + \\ & + (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mnp}\theta)a^{*p} + (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\omega^* \end{aligned} \tag{2.23}$$

where ω is the spacetime ghost, a_m and χ^μ are the fields, χ^*_μ and a^{*p} are the antifields, and ω^* is the antighost.

The cohomology of Q at non-zero momentum can be obtained by finding the constraints on these component fields implied by $Q\Psi = 0$ and $\delta\Psi = Q\Lambda$. One finds that ω and ω^* have trivial cohomology at non-zero momentum whereas a^{*p} and χ^*_μ satisfy the equations of motion and gauge invariances

$$\partial_p a^{*p} = 0\,, \qquad \delta a^*_m = \partial^n \partial_{[m}\sigma_{n]}\,, \qquad \delta\chi^*_\mu = \partial^q(\gamma_q\xi)_\mu\,. \tag{2.24}$$

As expected, the gauge invariances and equations of motion of the super-Maxwell antifields are related to the equations of motion and gauge invariances of the super-Maxwell fields of (2.21).

Using the wave function Ψ and the BRST operator $Q = \lambda^\mu d_\mu$, one can construct the spacetime action[6]

$$\mathcal{S} = \int d^{10}x \langle \Psi Q \Psi \rangle \tag{2.25}$$

where the norm $\langle\ \ \rangle$ is defined such that

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1\,. \tag{2.26}$$

Since $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ is the antighost state in (2.23) which cannot be written as $Q\Lambda$ for any Λ, the action of (2.25) is gauge invariant under $\delta\Psi = Q\Lambda$. Furthermore, the equations of motion from varying Ψ in (2.25) imply that $Q\Psi = 0$ for components in $Q\Psi$ involving up to five θ's. Although the manifestly supersymmetric equations of motion require that $Q\Psi = 0$ for all components of $Q\Psi$, one can check that any component of $Q\Psi = 0$ with more than five θ's is an auxiliary equation of motion which does not affect physical fields. So removing these auxiliary equations of motion changes auxiliary fields to gauge fields, but does not change the physical content of the theory.

[6] This action was first proposed to me by John Schwarz and Edward Witten [26] and generalizes to the super-Yang-Mills action $\mathcal{S} = \frac{1}{g^2}\int d^{10}x Tr\langle \Psi Q\Psi + \frac{2}{3}\Psi^3\rangle$. However, there does not appear to be a non-abelian generalization for the analogous action constructed using the $d = 11$ superparticle.

So the action of (2.25) reproduces the Batalin-Vilkovisky action for super-Maxwell theory and, if Ψ is restricted to ghost number one, (2.25) reproduces the standard super-Maxwell action. Note that the norm of (2.26) only involves integration over five of the sixteen θ^μ variables and therefore resembles a harmonic superspace. Since there are eleven independent bosonic λ^μ variables, one can interpret this integration over five θ's as coming from a cancellation between the λ^μ integration and the integration over eleven of the sixteen θ^μ variables.

2.4 Coupling the superparticle to a super-Maxwell background

To couple the pure spinor version of the superparticle action of (2.9) to a super-Maxwell background in a BRST-invariant manner, it is convenient to use the Oda-Tonin method[7] of [28] in which one first computes the BRST variation of the standard superparticle action in a super-Maxwell background of (2.7). Under the BRST transformation generated by $Q = \lambda^\mu d_\mu$,

$$Q\theta^\mu = \lambda^\mu \,, \qquad Qx^m = \frac{i}{2}\lambda\gamma^m\theta \,, \qquad Qd_\mu = -i\Pi^m(\gamma_m\lambda)_\mu \,, \qquad Qw_\mu = d_\mu \,, \qquad (2.27)$$

where the auxiliary equation of motion $P_m = \Pi_m$ has been used. One can check that Q^2 annihilates all variables except for w_μ, which satisfies $Q^2 w_\mu = -i\Pi_m(\gamma^m\lambda)_\mu$. This is consistent with the nilpotency of Q since $\delta w_\mu = -i\Pi_m(\gamma^m\lambda)_\mu$ is a gauge transformation of (2.11) with gauge parameter $\Lambda_m = -i\Pi_m$.

After fixing the reparameterization gauge $e = -1/2$ and using the auxiliary equation of motion for P_m, the standard superparticle action in a super-Maxwell background of (2.7) transforms under (2.27) as

$$Q\widehat{S} = i\int d\tau(\dot{\theta}^\mu - iqW^\mu)(\gamma^m\lambda)_\mu\Pi_m \qquad (2.28)$$

when the background superfields are on-shell. So if one adds the term

$$S' = \int d\tau\; Q[(\dot{\theta}^\mu - iqW^\mu)w_\mu] \qquad (2.29)$$

to the standard action,

$$Q(\widehat{S} + S') = Q\widehat{S} + \int d\tau\; Q^2[(\dot{\theta}^\mu - iqW^\mu)w_\mu] = Q\widehat{S} + \int d\tau(\dot{\theta}^\mu - iqW^\mu)Q^2 w_\mu = 0 \,. \quad (2.30)$$

[7] As was recently shown in [27] for the superparticle, this method for deriving the pure spinor action provides a gauge-fixing procedure for the Brink-Schwarz action of (2.7). One first gauge-fixes the Brink-Schwarz action using the standard BRST operator $Q_0 = cP^2$ together with the semi-light-cone choice $(\gamma^+\theta)^{\dot a} = 0$ and $\sqrt{P^+}(\gamma^-\theta)^a = S^a$. One then adds a new set of (θ^μ, p_μ) variables and unconstrained spinor ghost variables $(\widehat{\lambda}^\mu, \widehat{w}_\mu)$ to the Brink-Schwarz action, together with a topological term in the BRST operator, $Q = Q_0 + \widehat{\lambda}^\mu \widehat{d}_\mu$. Finally, one gauge-fixes $S^a = b = \widehat{\lambda}\gamma^m\widehat{\lambda} = 0$ to obtain the BRST operator $Q = \lambda^\mu d_\mu$ together with the pure spinor action of (2.31). Although this derivation of the pure spinor action has only been done for the superparticle [27], one expects that this method can also be generalized to the superstring and supermembrane.

Therefore, the BRST invariant action for the $N = 1$ $d = 10$ superparticle in a super-Maxwell background is

$$\begin{aligned} \widehat{S}_{\text{pure}} &= \widehat{S} + S' \\ &= \int d\tau \left[\frac{1}{2}\Pi^m \Pi_m + d_\mu \dot{\theta}^\mu + w_\mu \dot{\lambda}^\mu + q\Big(\dot{\theta}^\mu A_\mu + \Pi^m A_m - i d_\mu W^\mu - i(\lambda\gamma^{mn} w)F_{mn}\Big)\right] \end{aligned} \tag{2.31}$$

where $QW^\mu = \lambda^\nu D_\nu W = \lambda^\nu(\gamma^{mn})_\nu{}^\mu F_{mn}$ and F_{mn} is the super-Maxwell superfield whose lowest component is the vector field strength.

Furthermore, one can check that the integrand of the super-Maxwell interaction term,

$$V = \dot{\theta}^\mu A_\mu + \Pi^m A_m - i d_\mu W^\mu - i(\lambda\gamma^{mn} w)F_{mn}\,, \tag{2.32}$$

satisfies $QV = \frac{d}{d\tau}(\lambda^\mu A_\mu)$, which is the expected relation between the integrated super-Maxwell vertex operator and the unintegrated ghost number one vertex operator $\Psi = \lambda^\mu A_\mu$. Open superstring massless tree amplitudes can be computed in a manifestly super-Poincaré covariant manner by evaluating the correlation function of these integrated and unintegrated super-Maxwell vertex operators on the one-dimensional boundary of an open superstring worldsheet. By taking the tension of the string to infinity, these amplitudes reduce to $N = 1$ $d = 10$ supersymmetric Born-Infeld scattering amplitudes. To compute an N-point tree amplitude, one needs three unintegrated vertex operators and $N-3$ integrated vertex operators, and the normalization for the zero modes is defined as in (2.26) which, as desired, is non-vanishing for ghost number three. Furthermore, since the ghost number three antighost state of (2.26) is in the cohomology of Q and is not the supersymmetric variation of any state in the cohomology of Q, this normalization definition is manifestly gauge invariant and supersymmetric.

3. Covariant quantization of the $N = 2$ $d = 10$ superparticle

Before quantizing the $d = 11$ superparticle, it will be useful to discuss the $N = 2$ $d = 10$ superparticle. Since the $N = 2$ $d = 10$ superparticle describes the zero modes of the type-II superstring, its quantization is expected to describe a linearized version of type-II supergravity.

As will be seen in this section, there are subtleties at zero momentum with quantizing the $N = 2$ $d = 10$ superparticle which are related to subtleties with quantizing the type-II superstring. Recall that there is a left and right-moving set of (b_L, c_L) and (b_R, c_R) ghosts in closed string theory, and states in the closed string cohomology are required to be annihilated by both $Q_L + Q_R$ and by the zero mode $b_L^0 - b_R^0$. The absence of reparameterization ghosts in the pure spinor formalism makes it difficult to impose the $b_L^0 - b_R^0$ condition, which is related to the difficulty in constructing a kinetic term for closed superstring field theory. Remarkably, these subtleties will be resolved for the type-IIA superparticle in section 4 by taking the $P_{11} = 0$ limit of the $d = 11$ superparticle. The fact that these subtleties are not resolved for the type-IIB superparticle is not surprising because of the self-dual five-form field strength in the type-IIB supergravity spectrum.

3.1 Standard description of the $N = 2$ $d = 10$ superparticle

The standard action for the $N = 2$ $d = 10$ superparticle is

$$S = \int d\tau (P_m \Pi^m + eP^m P_m) \tag{3.1}$$

where

$$\Pi^m = \dot{x}^m + \frac{i}{2}\theta_L^\mu \gamma^m_{\mu\nu} \dot{\theta}_L^\nu + \frac{i}{2}\theta_R^{\hat{\mu}} \gamma^m_{\hat{\mu}\hat{\nu}} \dot{\theta}_R^{\hat{\nu}} , \tag{3.2}$$

$m = 1$ to 10, $\mu = 1$ to 16, $\hat{\mu} = 1$ to 16, and $(\theta_L^\mu, \theta_R^{\hat{\mu}})$ are the type-II fermionic superspace variables. For the type-IIA superparticle, μ and $\hat{\mu}$ denote spinors of opposite chirality, while for the type-IIB superparticle, μ and $\hat{\mu}$ denote spinors of the same chirality.

The action of (3.1) is invariant under the global $N = 2$ $d = 10$ spacetime-supersymmetry transformations

$$\delta\theta_L^\mu = \epsilon_L^\mu , \qquad \delta\theta_R^{\hat{\mu}} = \epsilon_R^{\hat{\mu}} , \qquad \delta x^m = \frac{i}{2}(\theta_L \gamma^m \epsilon_L + \theta_R \gamma^m \epsilon_R) , \qquad \delta P_m = \delta e = 0 , \tag{3.3}$$

and under the local kappa transformations

$$\begin{aligned} \delta\theta_L^\mu &= P^m(\gamma_m \kappa_L)^\mu , \qquad \delta\theta_R^{\hat{\mu}} = P^m(\gamma_m \kappa_R)^{\hat{\mu}} , \qquad \delta x^m = -\frac{i}{2}(\theta_L \gamma^m \delta\theta_L + \theta_R \gamma^m \delta\theta_R) , \\ \delta P_m &= 0 , \qquad \delta e = i(\dot{\theta}_L^\nu \kappa_{L\nu} + \dot{\theta}_R^{\hat{\nu}} \kappa_{R\hat{\nu}}) . \end{aligned} \tag{3.4}$$

The canonical momenta to θ_L^μ and $\theta_R^{\hat{\mu}}$, which will be called $p_{L\mu}$ and $p_{R\hat{\mu}}$, satisfy

$$p_{L\mu} = \frac{\partial L}{\partial \dot{\theta}_L^\mu} = \frac{i}{2} P^m (\gamma_m \theta_L)_\mu , \qquad p_{R\hat{\mu}} = \frac{\partial L}{\partial \dot{\theta}_R^{\hat{\mu}}} = \frac{i}{2} P^m (\gamma_m \theta_R)_{\hat{\mu}} ,$$

so canonical quantization requires that physical states are annihilated by the 32 fermionic Dirac constraints defined by

$$d_{L\mu} = p_{L\mu} - \frac{i}{2} P_m (\gamma^m \theta_L)_\mu , \qquad d_{R\hat{\mu}} = p_{R\hat{\mu}} - \frac{i}{2} P_m (\gamma^m \theta_R)_{\hat{\mu}} . \tag{3.5}$$

Since $\{p_{L\mu}, \theta_L^\nu\} = -i\delta_\mu^\nu$ and $\{p_{R\hat{\mu}}, \theta_R^{\hat{\nu}}\} = -i\delta_{\hat{\mu}}^{\hat{\nu}}$, these constraints satisfy the Poisson brackets

$$\{d_{L\mu}, d_{L\nu}\} = -P_m \gamma^m_{\mu\nu} , \qquad \{d_{R\hat{\mu}}, d_{R\hat{\nu}}\} = -P_m \gamma^m_{\hat{\mu}\hat{\nu}} , \qquad \{d_{L\mu}, d_{R\hat{\nu}}\} = 0 , \tag{3.6}$$

and since $P^m P_m = 0$ is also a constraint, 16 of the 32 Dirac constraints are first-class and 16 are second-class. One can easily check that the 16 first-class Dirac constraints generate the kappa transformations of (3.4), however, there is no simple way to covariantly separate out the second-class constraints.

3.2 Pure spinor description of the $N = 2$ $d = 10$ superparticle

Instead of using the standard $N = 2$ superparticle action of (3.1), the pure spinor formalism for the $N = 2$ $d = 10$ superparticle uses the quadratic action

$$S_{\text{pure}} = \int d\tau \left(P_m \dot{x}^m + p_{L\mu} \dot{\theta}_L^\mu + p_{R\hat{\mu}} \dot{\theta}_R^{\hat{\mu}} + w_{L\mu} \dot{\lambda}_L^\mu + w_{R\hat{\mu}} \dot{\lambda}_R^{\hat{\mu}} - \frac{1}{2} P^m P_m \right) \tag{3.7}$$

where $p_{L\mu}$ and $p_{R\hat{\mu}}$ are now independent variables, λ_L^μ and $\lambda_R^{\hat{\mu}}$ are pure spinor ghost variables satisfying

$$\lambda_L \gamma^m \lambda_L = 0 \qquad \text{and} \qquad \lambda_R \gamma^m \lambda_R = 0 \qquad \text{for } m = 1 \text{ to } 10\,, \tag{3.8}$$

and $w_{L\mu}$ and $w_{R\hat{\mu}}$ are defined up to the gauge transformations

$$\delta w_{L\mu} = (\gamma^m \lambda_L)_\mu \Lambda_{Lm}\,, \qquad \delta w_{R\hat{\mu}} = (\gamma^m \lambda_R)_{\hat{\mu}} \Lambda_{Rm}\,, \tag{3.9}$$

for arbitrary gauge parameters Λ_{Lm} and Λ_{Rm}. The action of (3.7) can be written in manifestly spacetime supersymmetric notation as

$$S_{\text{pure}} = \int d\tau \left(P_m \Pi^m + d_{L\mu} \dot{\theta}_L^\mu + d_{R\hat{\mu}} \dot{\theta}_R^{\hat{\mu}} + w_{L\mu} \dot{\lambda}_L^\mu + w_{R\hat{\mu}} \dot{\lambda}_R^{\hat{\mu}} - \frac{1}{2} P^m P_m \right) \tag{3.10}$$

where Π^m, $d_{L\mu}$ and $d_{R\hat{\mu}}$ are defined as in (3.2) and (3.5).

To obtain the correct physical spectrum, the action of (3.7) will be supplemented with the constraint that physical states are in the cohomology of the left and right-moving BRST-like operators

$$Q_L = \lambda_L^\mu d_{L\mu} \qquad \text{and} \qquad Q_R = \lambda_R^{\hat{\mu}} d_{R\hat{\mu}}\,. \tag{3.11}$$

In other words, physical states Ψ will be defined by the equations of motion and gauge invariances

$$Q_L \Psi = Q_R \Psi = 0\,, \qquad \delta\Psi = Q_L \Lambda_L + Q_R \Lambda_R\,, \tag{3.12}$$

where the gauge parameters Λ_L and Λ_R are constrained to satisfy $Q_R \Lambda_L = Q_L \Lambda_R = 0$. As will now be shown, this definition of physical states at (left,right) ghost number $(1,1)$ and non-zero momentum reproduces the correct linearized type-II supergravity spectrum. However, at zero momentum, the definition of (3.12) omits certain states in the supergravity spectrum. As will be explained below, this is caused by the absence of the $N=2$ superparticle analog of the $b_L^0 - b_R^0$ constraint for the type-II superstring.

Note that in light-cone gauge, the 44 independent $(\lambda_L^\mu, \lambda_R^{\hat{\mu}})$ and $(w_{L\mu}, w_{R\hat{\mu}})$ bosonic ghost variables cancel 44 of the 64 fermionic $(\theta_L^\mu, \theta_R^{\hat{\mu}})$ and $(p_{L\mu}, p_{R\hat{\mu}})$ variables, leaving twenty fermionic variables. Two of these twenty fermionic variables act as the missing (b,c) reparameterization ghosts and cancel the longtitudinal components of x^m and P_m. However, besides the sixteen physical light-cone fermionic variables, there are still two extra fermionic variables which need to be eliminated. These two extra fermionic variables are the $N=2$ superparticle analog of the $(b_L - b_R, c_L - c_R)$ zero modes in the closed superstring.

3.3 BRST description of linearized type-II supergravity at non-zero momentum

Since Q_L and Q_R are constructed from independent variables, the physical states defined by (3.12) for the $N=2$ superparticle are described by the "left-right" product of two $N=1$ superparticle physical states. At (left,right) ghost number $(1,1)$, the $N=2$ superparticle wavefunction

$$\Psi(\lambda_L, \lambda_R, x, \theta_L, \theta_R) = \lambda_L^\mu \lambda_R^{\hat{\nu}} A_{\mu\hat{\nu}}(x, \theta_L, \theta_R) \tag{3.13}$$

is physical if $A_{\mu\hat{\nu}}$ satisfies

$$(\gamma_{mnpqr})^{\rho\mu} D_{L\rho} A_{\mu\hat{\nu}} = (\gamma_{mnpqr})^{\hat{\rho}\hat{\nu}} D_{R\hat{\rho}} A_{\mu\hat{\nu}} = 0 \tag{3.14}$$

with the gauge invariances

$$\delta A_{\mu\hat{\nu}} = D_{L\mu}\Lambda_{L\hat{\nu}} + D_{R\hat{\nu}}\Lambda_{R\mu} \qquad \text{where} \qquad (\gamma_{mnpqr})^{\hat{\rho}\hat{\mu}} D_{R\hat{\rho}}\Lambda_{L\hat{\mu}} = (\gamma_{mnpqr})^{\rho\mu} D_{L\rho}\Lambda_{R\mu} = 0\,, \tag{3.15}$$

and

$$D_{L\mu} = \frac{\partial}{\partial\theta_L^{\mu}} + \frac{i}{2}(\gamma^m\theta_L)_\mu\partial_m\,, \qquad D_{R\hat{\mu}} = \frac{\partial}{\partial\theta_R^{\hat{\mu}}} + \frac{i}{2}(\gamma^m\theta_R)_{\hat{\mu}}\partial_m\,, \tag{3.16}$$

are the $N = 2$ $d = 10$ supersymmetric derivatives.

In components, (3.14) and (3.15) imply that the physical states of the $N = 2$ superparticle are described by left-right product of super-Maxwell photons and photinos. That is, at ghost number $(1,1)$

$$\begin{aligned}\Psi(\lambda_L,\lambda_R,x,\theta_L,\theta_R) \;=\; & (\lambda_L\gamma^m\theta_L)(\lambda_R\gamma^n\theta_R)a_{mn}(x) + \\ & +(\lambda_L\gamma^m\theta_L)(\lambda_R\gamma^n\theta_R)(\theta_R\gamma_n)_{\hat{\mu}}\chi^{\hat{\mu}}_{Lm}(x) + \\ & +(\lambda_L\gamma^m\theta_L)(\theta_L\gamma_m)_\mu(\lambda_R\gamma^n\theta_R)\chi^{\mu}_{Rn}(x) + \\ & +(\lambda_L\gamma^m\theta_L)(\theta_L\gamma_m)_\mu(\lambda_R\gamma^n\theta_R)(\theta_R\gamma_n)_{\hat{\nu}}F^{\mu\hat{\nu}}(x) + \cdots\end{aligned} \tag{3.17}$$

where the higher components in $\ldots$ can be expressed in terms of a_{mn}, $\chi^{\hat{\mu}}_{Lm}$, χ^{μ}_{Rn} and $F^{\mu\hat{\nu}}$. Furthermore, these fields satisfy the equations of motion

$$\begin{aligned}& \partial^m(\partial_m a_{np} - \partial_n a_{mp}) = \partial^m(\partial_m a_{np} - \partial_p a_{nm}) = 0\,, \\ & \partial^m\partial_{[m}\chi^{\hat{\mu}}_{Ln]} = \gamma^m_{\hat{\mu}\hat{\nu}}\partial_m\chi^{\hat{\nu}}_{Ln} = 0\,, \qquad \partial^m\partial_{[m}\chi^{\mu}_{Rn]} = \gamma^m_{\mu\nu}\partial_m\chi^{\nu}_{Rn} = 0\,, \\ & \gamma^m_{\hat{\mu}\hat{\nu}}\partial_m F^{\rho\hat{\nu}} = \gamma^m_{\mu\nu}\partial_m F^{\nu\hat{\rho}} = 0\,,\end{aligned} \tag{3.18}$$

and gauge invariances

$$\delta a_{mn} = \partial_m\omega_{Ln} + \partial_n\omega_{Rm}\,, \qquad \delta\chi^{\hat{\mu}}_{Lm} = \partial_m\sigma^{\hat{\mu}}_L\,, \qquad \delta\chi^{\mu}_{Rm} = \partial_m\sigma^{\mu}_R\,, \tag{3.19}$$

where the gauge parameters satisfy

$$\partial^m\partial_{[m}\omega_{Ln]} = \partial^m\partial_{[m}\omega_{Rn]} = \gamma^m_{\hat{\mu}\hat{\nu}}\partial_m\sigma^{\hat{\nu}}_L = \gamma^m_{\mu\nu}\partial_m\sigma^{\nu}_R = 0\,. \tag{3.20}$$

If one chooses the Lorentz gauge

$$\partial^m a_{mn} = \partial^n a_{mn} = \partial^m\chi^{\hat{\mu}}_{Lm} = \partial^m\chi^{\mu}_{Rm} = 0\,, \tag{3.21}$$

the equations of motion and gauge invariances of (3.18) and (3.19) are those of linearized type-II supergravity where $a_{mn} = h_{mn} + b_{mn} + \eta_{mn}\phi$ describes the symmetric traceless graviton h_{mn}, antisymmetric two-form b_{mn} and dilaton ϕ, where $\chi^{\hat{\mu}}_{Lm} = \rho^{\hat{\mu}}_{Lm} + \gamma^{\hat{\mu}\hat{\nu}}_m\xi_{L\hat{\nu}}$ and $\chi^{\mu}_{Rm} = \rho^{\mu}_{Rm} + \gamma^{\mu\nu}_m\xi_{R\nu}$ describe the $N = 2$ gamma-matrix traceless gravitini $[\rho^{\hat{\mu}}_{Lm}, \rho^{\mu}_{Rm}]$ and dilatini $[\xi_{L\hat{\nu}}, \xi_{R\nu}]$, and where $F^{\mu\hat{\nu}}$ describes the Ramond-Ramond field strengths.

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So at non-zero momentum, where Lorentz gauge is possible, the ghost number $(1,1)$ fields in Ψ correctly describe the linearized type-II supergravity fields. However, at zero momentum, not all the physical type-II supergravity fields are included in $(a_{mn}, \chi^{\hat\mu}_{Lm}, \chi^{\mu}_{Rm}, F^{\mu\hat\nu})$. For example, both the dilaton and the trace of the metric are physical scalars at zero momentum, but a_{mn} only contains one scalar. Similarly, the Ramond-Ramond gauge fields at zero momentum are not described by $F^{\mu\hat\nu}$. The absence of these zero momentum fields prevents the construction of a type-II supergravity kinetic term which would be the $N = 2$ superparticle analog of the super-Maxwell action constructed in (2.25).

As mentioned earlier, this problem is related to the absence of the $b^0_L - b^0_R$ constraint in the pure spinor formalism. In closed string field theory, the correct definition of physical states uses the BRST cohomology of $Q = Q_L + Q_R$, together with the constraint that states are annihilated by $b^0_L - b^0_R$. Although this definition agrees with (3.12) at non-zero momentum, it does not agree with (3.12) at zero momentum. Although it will not be possible to impose the $b^0_L - b^0_R$ constraint for the $N = 2$ $d = 10$ superparticle, it will now be shown that this constraint is naturally imposed when one quantizes the $d = 11$ superparticle using pure spinors.

4. Covariant quantization of the $d = 11$ superparticle

In this section, the $d = 11$ superparticle will be covariantly quantized in a manner which allows a BRST description of linearized $d = 11$ supergravity.

4.1 Standard description of the $d = 11$ superparticle

The standard action for the $d = 11$ superparticle is

$$S = \int d\tau (P_c \Pi^c + e P^c P_c) \tag{4.1}$$

where

$$\Pi^c = \dot x^c + \frac{i}{2}\theta^\alpha \Gamma^c_{\alpha\beta}\dot\theta^\beta\,, \tag{4.2}$$

$c = 1$ to 11 is the $\mathrm{SO}(10,1)$ vector index with x^{10} as the time coordinate, and $\alpha = 1$ to 32 is the $\mathrm{SO}(10,1)$ spinor index. The $d = 11$ gamma matrices $\Gamma^c_{\alpha\beta}$ are 32×32 symmetric matrices which satisfy $\Gamma^{(c}_{\alpha\beta}\Gamma^{d)\ \beta\gamma} = 2\eta^{cd}\delta^\gamma_\alpha$. In $d = 11$, spinor indices can be raised and lowered using the antisymmetric metric tensor $C^{\alpha\beta}$ and its inverse $C^{-1}_{\alpha\beta}$. For example, $\Gamma^{c\alpha\beta} = C^{\alpha\delta}\Gamma^{c\beta}_{\delta} = C^{\alpha\delta}C^{\beta\gamma}\Gamma^c_{\delta\gamma}$. Note that any $d = 11$ antisymmetric bispinor $f^{[\alpha\beta]}$ can be decomposed into a scalar, three-form, and four-form as $f^{[\alpha\beta]} = C^{\alpha\beta} f + (\Gamma_{bcd})^{\alpha\beta} f^{bcd} + (\Gamma_{bcde})^{\alpha\beta} f^{bcde}$, and any $d = 11$ symmetric bispinor $g^{(\alpha\beta)}$ can be decomposed into a one-form, two-form and five-form as $g^{(\alpha\beta)} = \Gamma^{\alpha\beta}_c g^c + (\Gamma_{cd})^{\alpha\beta} g^{cd} + (\Gamma_{bcdef})^{\alpha\beta} g^{bcdef}$. Furthermore, the $d = 11$ gamma matrices satisfy the identity $\eta_{bc}\Gamma^b_{(\alpha\beta}\Gamma^{cd}_{\gamma\delta)} = 0$.

The action of (4.1) is invariant under the global $d = 11$ spacetime-supersymmetry transformations

$$\delta\theta^\alpha = \epsilon^\alpha\,, \qquad \delta x^c = \frac{i}{2}\theta\Gamma^c\epsilon\,, \qquad \delta P_c = \delta e = 0\,, \tag{4.3}$$

and under the local kappa transformations

$$\delta\theta^\alpha = P^c(\Gamma_c\kappa)^\alpha\,, \qquad \delta x^c = -\frac{i}{2}\theta\Gamma^c\delta\theta\,, \qquad \delta P_c = 0\,, \qquad \delta e = i\dot\theta^\alpha\kappa_\alpha\,. \tag{4.4}$$

The canonical momentum to θ^α, which will be called p_α, satisfies

$$p_\alpha = \frac{\partial L}{\partial\dot\theta^\alpha} = \frac{i}{2}P^c(\Gamma_c\theta)_\alpha\,,$$

so canonical quantization requires that physical states are annihilated by the 32 fermionic Dirac constraints defined by

$$d_\alpha = p_\alpha - \frac{i}{2}P_c(\Gamma^c\theta)_\alpha\,. \tag{4.5}$$

Since $\{p_\alpha,\theta^\beta\} = -i\delta_\alpha^\beta$, these constraints satisfy the Poisson brackets

$$\{d_\alpha, d_\beta\} = -P_c\Gamma^c_{\alpha\beta}\,, \tag{4.6}$$

and since $P^cP_c = 0$ is also a constraint, 16 of the 32 Dirac constraints are first-class and 16 are second-class. One can easily check that the 16 first-class Dirac constraints generate the kappa transformations of (4.4), however, there is no simple way to covariantly separate out the second-class constraints.

4.2 Pure spinor description of the $d = 11$ superparticle

At $P_{11} = 0$, the action of (4.1) reduces to the standard type-IIA $N = 2$ superparticle action of (3.1) where $\theta_L^\mu = \frac{1}{\sqrt2}(1+\Gamma^{11})^\mu_\alpha\theta^\alpha$ and $\theta_R^{\hat\mu} = \frac{1}{\sqrt2}(1-\Gamma^{11})^{\hat\mu}_\alpha\theta^\alpha$. This suggests constructing a new pure spinor version of the $d = 11$ superparticle action which instead reduces at $P_{11} = 0$ to the type-IIA $N = 2$ superparticle action of (3.7). This pure spinor version of the $d = 11$ superparticle action will be defined as the quadratic action

$$S_{\rm pure} = \int d\tau\left(P_c\dot x^c + p_\alpha\dot\theta^\alpha + w_\alpha\dot\lambda^\alpha - \frac{1}{2}P^cP_c\right) \tag{4.7}$$

where p_α is an independent variable, λ^α is an SO(10, 1) pure spinor ghost variable satisfying

$$\lambda\Gamma^c\lambda = 0 \qquad \text{for } c = 1 \text{ to } 11\,, \tag{4.8}$$

and w_α is the canonical momentum to λ^α which is defined up to the gauge transformation

$$\delta w_\alpha = (\Gamma^c\lambda)_\alpha\Lambda_c \tag{4.9}$$

for arbitrary gauge parameter Λ_c.

With the exception of the $d = 11$ pure spinor constraint of (4.8), the action of (4.7) reduces when $P_{11} = 0$ to the type-IIA $N = 2$ superparticle action of (3.7) where

$$\begin{aligned}
\theta_L^\mu &= \frac{1}{\sqrt2}(1+\Gamma^{11})^\mu_\alpha\theta^\alpha\,, & \theta_R^{\hat\mu} &= \frac{1}{\sqrt2}(1-\Gamma^{11})^{\hat\mu}_\alpha\theta^\alpha\,,\\
p_{L\mu} &= \frac{1}{\sqrt2}(1-\Gamma^{11})_\mu^\alpha p_\alpha\,, & p_{R\hat\mu} &= \frac{1}{\sqrt2}(1+\Gamma^{11})_{\hat\mu}^\alpha p_\alpha\,,\\
\lambda_L^\mu &= \frac{1}{\sqrt2}(1+\Gamma^{11})^\mu_\alpha\lambda^\alpha\,, & \lambda_R^{\hat\mu} &= \frac{1}{\sqrt2}(1-\Gamma^{11})^{\hat\mu}_\alpha\lambda^\alpha\,,\\
w_{L\mu} &= \frac{1}{\sqrt2}(1-\Gamma^{11})_\mu^\alpha w_\alpha\,, & w_{R\hat\mu} &= \frac{1}{\sqrt2}(1+\Gamma^{11})_{\hat\mu}^\alpha w_\alpha\,.
\end{aligned} \tag{4.10}$$

However, the $d=11$ pure spinor constraint $\lambda\Gamma^c\lambda = 0$ does not reduce to the $N=2$ $d=10$ pure spinor constraints $\lambda_L\gamma^m\lambda_L = \lambda_R\gamma^m\lambda_R = 0$ of (3.8). As will now be shown, the difference between these constraints resolves the difficulties discussed in the previous subsection for quantization of the type-IIA $N=2$ superparticle.

After decomposing λ^α into λ_L^μ and $\lambda_{R\mu}$, the constraint $\lambda\Gamma^c\lambda = 0$ for $c=1$ to 11 implies that

$$\begin{aligned} \lambda\Gamma^m\lambda = \lambda_L\gamma^m\lambda_L + \lambda_R\gamma^m\lambda_R &= 0 \qquad \text{for } m = 1 \text{ to } 10\,, \\ \lambda\Gamma^{11}\lambda = \lambda_L^\mu\lambda_{R\mu} &= 0\,. \end{aligned} \tag{4.11}$$

The first line of (4.11) is obviously satisfied when $\lambda_L\gamma^m\lambda_L = \lambda_R\gamma^m\lambda_R = 0$, however, the second line is new and is not implied by (3.8).

Note that when $\lambda\Gamma^{11}\lambda$ is non-zero, the constraint $\lambda\Gamma^m\lambda = 0$ implies that

$$\lambda\Gamma^m\Gamma^{11}\lambda = \lambda_L\gamma^m\lambda_L - \lambda_R\gamma^m\lambda_R = 0 \qquad \text{for } m = 1 \text{ to } 10\,, \tag{4.12}$$

which are the remaining constraints of (3.8). To prove this, use the $d=11$ identity $\eta_{de}\Gamma^{cd}_{(\alpha\beta}\Gamma^e_{\gamma\delta)} = 0$ to argue that

$$(\lambda\Gamma^{c\ 11}\lambda)(\lambda\Gamma^{11}\lambda) = -(\lambda\Gamma^{cm}\lambda)(\lambda\Gamma_m\lambda)\,. \tag{4.13}$$

So when $\lambda\Gamma^m\lambda = 0$, either $\lambda\Gamma^{c\ 11}\lambda$ or $\lambda\Gamma^{11}\lambda$ must vanish. In the $N=2$ $d=10$ superparticle of the previous subsection, $\lambda\Gamma^{c\ 11}\lambda$ was constrained to vanish. However, in the $d=11$ superparticle, $\lambda\Gamma^{11}\lambda$ will be constrained to vanish.

To compute the number of independent degrees of freedom of a $d=11$ pure spinor λ^α satisfying (4.8), note that $\eta_{mn}\gamma^m_{(\mu\nu}\gamma^n_{\kappa)\rho} = 0$ implies that $h^m = \lambda_L\gamma^m\lambda_L$ satisfies $h^m h_m = 0$, and suppose that h^m is non-vanishing. Then the first line of (4.11) implies that $\lambda_R\gamma^m\lambda_R = -h^m$, which constrains nine of the sixteen $\lambda_{R\mu}$ variables, leaving seven independent variables for $\lambda_{R\mu}$. Furthermore, using the argument of the previous paragraph and assuming that h^m is non-vanishing, the second line of (4.11) imposes no further constraints. So λ^α contains 23 independent degrees of freedom, sixteen coming from λ_L^μ and seven coming from $\lambda_{R\mu}$.[8]

Replacing the $N=2$ $d=10$ pure spinor constraint of (3.8) with the $d=11$ pure spinor constraint of (4.8), one can check that the light-cone counting of degrees of freedom now gives the correct answer. After using the 46 independent λ^α and w_α bosonic ghost variables to cancel 46 of the 64 fermionic θ^α and p_α variables, one is left with 18 fermionic variables. Two of these fermionic variables replace the missing (b,c) reparameterization ghosts and the remaining 16 variables are the physical light-cone fermionic variables. So the $\lambda\Gamma^{11}\lambda = 0$ condition has effectively replaced the $b_L^0 - b_R^0$ constraint of the closed superstring.

Physical states for the $d=11$ superparticle will be defined as states in the cohomology of the BRST-like operator

$$Q = \lambda^\alpha d_\alpha\,, \tag{4.14}$$

[8] As in the $d=10$ case, the $d=11$ pure spinor λ^α must be complex to satisfy (4.8), but its complex conjugate $\overline{\lambda^\alpha}$ never appears in the formalism and can therefore be ignored. Note that if Howe's definition of $d=11$ pure spinors of [17] had been used, λ^α would have had sixteen independent components.

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which is nilpotent because of (4.6) and (4.8). As will now be shown, this definition of physical states correctly describes the spacetime fields of linearized $d = 11$ supergravity, as well as describing the spacetime ghosts, antifields and antighosts of the theory. Note that Q of (4.14) reduces at $P_{11} = 0$ to the sum of the $N = 2$ $d = 10$ BRST operators $Q = Q_R + Q_L = \lambda_L^\mu d_{L\mu} + \lambda_{R\mu} d_R^\mu$. However, using the $d = 11$ superparticle quantization, the linearized $d = 11$ supergravity states will reduce at $P_{11} = 0$ to the linearized type-IIA supergravity states without any of the problems at zero momentum encountered in the previous section using the $N = 2$ superparticle quantization.

4.3 BRST description of linearized $d = 11$ supergravity

As opposed to ghost-number one physical fields of the $N = 1$ $d = 10$ superparticle and ghost-number two physical fields of the $N = 2$ $d = 10$ superparticle, the physical fields of the $d = 11$ superparticle will appear at ghost-number three in the cohomology of Q. As will be seen in subsection 7.1, this comes from the fact that $d = 10$ super-Maxwell theory couples to the one-dimensional worldline of the superparticle, type-II supergravity couples to the two-dimensional worldsheet of the closed superstring, and $d = 11$ supergravity couples to the three-dimensional worldvolume of the supermembrane.

At ghost-number three, the $d = 11$ superparticle wavefunction is

$$\Psi(\lambda, x, \theta) = \lambda^\alpha \lambda^\beta \lambda^\gamma A_{\alpha\beta\gamma}(x, \theta) \tag{4.15}$$

where $A_{\alpha\beta\gamma}$ is an arbitrary $d = 11$ superfield which is symmetric in its spinor indices and which, because of (4.8), is defined up to the algebraic gauge transformation $\delta A_{\alpha\beta\gamma} = \gamma^c_{(\alpha\beta} F_{\gamma)c}$ for arbitrary $F_{\gamma c}$. The equation of motion $Q\Psi{=}0$ implies that $\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta D_\alpha A_{\beta\gamma\delta} = 0$, and $\delta\Psi = Q\Lambda$ implies the gauge transformation $\delta A_{\beta\gamma\delta} = D_{(\beta}\Lambda_{\gamma\delta)}$ where $\Lambda = \lambda^\alpha\lambda^\beta\Lambda_{\alpha\beta}$ and $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}(\gamma^c\theta)_\alpha\partial_c$ is the $d = 11$ supersymmetric derivative. It will now be shown that these equations of motion and gauge invariances describe the linearized $d = 11$ supergravity fields. In fact, up to a gauge transformation and with an appropriate choice of conventional constraints, $A_{\alpha\beta\gamma}$ is expected[9] to be the linearized spinor component $B_{\alpha\beta\gamma}$ of the three-form superfield B_{ABC} of $d = 11$ supergravity.

Expanding in components, one can show that Ψ of (4.15) can be gauge-fixed to the form

$$\begin{aligned}\Psi(\lambda, x, \theta) &= \lambda^\alpha\lambda^\beta\lambda^\gamma A_{\alpha\beta\gamma}(x, \theta)\\ &= (\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta) b_{abc}(x) + (\lambda\gamma^{(a}\theta)(\lambda\gamma^{b)c}\theta)(\lambda\gamma_c\theta) g_{ab}(x) +\\ &\quad + (\lambda\gamma^b\theta)[(\lambda\gamma^c\theta)(\lambda\gamma^d\theta)(\theta\gamma_{cd})_\alpha - (\lambda\gamma^{cd}\theta)(\lambda\gamma_c\theta)(\theta\gamma_d)_\alpha]\chi_b^\alpha(x) + \cdots\end{aligned} \tag{4.16}$$

where terms in $\cdots$ involve more than four θ's and can be expressed in terms of $b_{abc}(x)$, $g_{ab}(x)$ and $\chi_b^\alpha(x)$. Note that b_{abc} is antisymmetric in its indices and g_{ab} is symmetric in its indices. Furthermore, the component fields $b_{abc}(x)$, $g_{ab}(x)$ and $\chi_b^\alpha(x)$ satisfy the equations

[9] I would like to thank Paul Howe for suggesting that $B_{\alpha\beta\gamma}$ might play such a role. This suggestion was independently confirmed in reference [20].

of motion and gauge invariances[10]

$$\begin{aligned} \partial^d \partial_{[a} b_{bcd]} &= 0\,, & \delta b_{abc} &= \partial_{[a}\omega_{bc]}\,, \\ \partial^a(\partial_a g_{bc} - 2\partial_{(b} g_{c)a}) + \partial_b\partial_c(\eta^{de} g_{de}) &= 0\,, & \delta g_{ab} &= \partial_{(a}\rho_{b)}\,, \\ (\gamma^{abc})_{\alpha\beta}\partial_b \chi_c^\beta &= 0\,, & \delta\chi_a^\beta &= \partial_a \xi^\beta\,, \end{aligned} \tag{4.17}$$

which identify them as the linearized three-form, graviton and gravitino of $d = 11$ supergravity. So the ghost-number three cohomology of Q correctly describes the linearized $d = 11$ supergravity fields without any subtleties at zero momentum.

To show that the cohomology of Q at other ghost numbers correctly describes the ghosts, antifields and antighosts of linearized $d = 11$ supergravity, it is convenient to first compute the zero momentum cohomology using a BRST operator $\widetilde{Q}$ with an unconstrained $d = 11$ spinor $\widetilde{\lambda}^\alpha$. As discussed in the appendix of this paper, the cohomology of Q at zero momentum is equivalent to the cohomology of

$$\widetilde{Q} = \widetilde{\lambda}^\alpha p_\alpha + \widetilde{\lambda}\Gamma^c\widetilde{\lambda} b_{(-1)c} + c^c_{(1)}\left(\widetilde{\lambda}\Gamma_{cd}\widetilde{\lambda} u^d_{(-2)} + \widetilde{\lambda}\Gamma^d\widetilde{\lambda} u_{(-2)[cd]}\right) + \cdots \tag{4.18}$$

where $[b_{(-n)}, c_{(n)}]$ and $[u_{(-n)}, v_{(n)}]$ are new fermionic and bosonic pairs of conjugate variables of ghost number $[-n, n]$ and $\cdots$ involves ghost numbers up to $[-7, 7]$ whose explicit form can be found in the appendix. As in the discussion of subsection 2.3 for the $N = 1$ $d = 10$ superparticle, the term $\widetilde{\lambda}\Gamma^c\widetilde{\lambda} b_{(-1)c}$ in (4.18) imposes the pure spinor constraint and the terms $c^c_{(1)}(\widetilde{\lambda}\Gamma_{cd}\widetilde{\lambda} u^d_{(-2)} + \widetilde{\lambda}\Gamma^d\widetilde{\lambda} u_{(-2)[cd]}) + \cdots$ remove the extra gauge invariances implied by this constraint.

As will be discussed in the appendix, the states in the cohomology of $\widetilde{Q}$ are in one-to-one correspondence with the variables

$$\left[1, c^c_{(1)}, v^c_{(2)}, v^{[cd]}_{(2)}, c^\alpha_{(2)}, c^{(cd)}_{(3)}, c^{[cde]}_{(3)}, v^{c\alpha}_{(3)}, c^{c\alpha}_{(4)}, v^{[cde]}_{(4)}, v^{(cd)}_{(4)}, v^\alpha_{(5)}, c^{[cd]}_{(5)}, c^c_{(5)}, v^c_{(6)}, c_{(7)}\right]. \tag{4.19}$$

The ghost number three states corresponding to $[c^{(cd)}_{(3)}, c^{[cde]}_{(3)}, v^{c\alpha}_{(3)}]$ are the graviton, antisymmetric three-form, and gravitino fields of linearized $d = 11$ supergravity, and the ghost number four states corresponding to $[v^{(cd)}_{(4)}, v^{[cde]}_{(4)}, c^{c\alpha}_{(4)}]$ are their antifields. The ghost number two states corresponding to $[v^c_{(2)}, c^\alpha_{(2)}, v^{[cd]}_{(2)}]$ are the ghosts coming from super-reparameterization invariance and the three-form gauge invariance $\delta b_{cde} = \partial_{[c}\Lambda_{de]}$, and the ghost number five states corresponding to $[c^c_{(5)}, v^\alpha_{(5)}, c^{[cd]}_{(5)}]$ are their antighosts. The ghost number one state corresponding to $c^c_{(1)}$ is the ghost-for-ghost coming from the gauge invariance of the two-form gauge parameter $\delta\Lambda_{de} = \partial_{[d}\Lambda'_{e]}$, and the ghost number six state corresponding to $v^c_{(6)}$ is its antighost-for-antighost. Finally, the ghost number zero state corresponding to 1 is the ghost-for-ghost-for-ghost coming from the gauge invariance of the ghost-for-ghost gauge parameter $\delta\Lambda'_e = \partial_e\Lambda''$, and the ghost number seven state corresponding to $c_{(7)}$ is its antighost-for-antighost-for-antighost. So using the results of the appendix, the zero momentum cohomology of Q correctly describes the zero momentum ghosts, fields, antifields and antighosts of linearized $d = 11$ supergravity.

[10] Although one can in principle derive these equations of motion directly from the higher θ components of $Q\Psi = 0$, they can be justified indirectly using the cohomology structure of the antifields which will be discussed below.

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Using the one-to-one map between states at zero momentum in the cohomologies of Q and $\widetilde{Q}$, one finds that the wavefunction for the zero momentum states in the cohomology of Q with constrained λ^α is

$$\begin{aligned}\Psi(\lambda,\theta) = & \ \omega'' + (\lambda\theta)^c\omega'_c + (\lambda^2\theta^2)^{[cd]}\omega_{[cd]} + (\lambda^2\theta^2)^c\rho_c + (\lambda^2\theta^3)^\alpha\xi_\alpha + \\ & + (\lambda^3\theta^3)^{[cde]}b_{[cde]} + (\lambda^3\theta^3)^{(cd)}g_{(cd)} + (\lambda^3\theta^4)^{c\alpha}\chi_{c\alpha} + \\ & + (\lambda^4\theta^5)^{c\alpha}\chi^*_{c\alpha} + (\lambda^4\theta^6)^{(cd)}g^*_{(cd)} + (\lambda^4\theta^6)^{[cde]}b^*_{[cde]} + \\ & + (\lambda^5\theta^6)^\alpha\xi^*_\alpha + (\lambda^5\theta^7)^c\rho^*_c + (\lambda^5\theta^7)^{[cd]}\omega^*_{[cd]} + (\lambda^6\theta^8)^c\omega'^*_c + (\lambda^7\theta^9)\omega''^* , \qquad (4.20)\end{aligned}$$

where $[\omega_{[cd]},\omega'_c,\omega'']$ and $[\omega^*_{[cd]},\omega'^*_c,\omega''^*]$ are the ghosts and antighosts for the three-form gauge invariance, $[\rho_c,\xi_\alpha]$ and $[\rho^*_c,\xi^*_\alpha]$ are the ghosts and antighosts for the super-reparameterization invariance, $[b_{[cde]},g_{(cd)},\chi_{c\alpha}]$ and $[b^*_{[cde]},g^*_{(cd)},\chi^*_{c\alpha}]$ are the linearized $d=11$ supergravity fields and antifields, and to simplify notation, the contractions of the spinor indices of λ^α and θ^β in (4.20) have been suppressed. Note that the contractions of the spinor indices in the scalar ghost number seven state denoted $(\lambda^7\theta^9)$ can be determined indirectly using the fact that

$$(\lambda\Gamma^{c_1}\theta)(\lambda\Gamma^{c_2}\theta)\cdots(\lambda\Gamma^{c_9}\theta) = \epsilon^{c_1\cdots c_9 de}(\lambda\Gamma_{de}\lambda)(\lambda^7\theta^9), \qquad (4.21)$$

which can be proven using the identity $(\lambda\Gamma^c)_\alpha(\lambda\Gamma_{cd}\lambda)=0$. This is analogous to the fact that the ghost number three scalar state $(\lambda^3\theta^5)$ in the $N=1$ $d=10$ cohomology satisfies

$$(\lambda\gamma^{m_1}\theta)(\lambda\gamma^{m_2}\theta)\cdots(\lambda\gamma^{m_5}\theta) = \epsilon^{m_1\cdots m_5 npqrs}(\lambda\Gamma_{npqrs}\lambda)(\lambda^3\theta^5), \qquad (4.22)$$

which can be proven using the identity $(\lambda\gamma^m)_\mu(\lambda\gamma_{mnpqr}\lambda)=0$.

The cohomology of Q at non-zero momentum can be obtained by finding the constraints on the component fields of (4.20) implied by $Q\Psi=0$ and $\delta\Psi=Q\Lambda$. One finds that all ghosts and antighosts have trivial cohomology at non-zero momentum, the supergravity fields satisfy the equations of motion and gauge invariances of (4.17), and the supergravity antifields satisfy the equations of motion and gauge invariances

$$\begin{aligned} \partial^a b^*_{abc} &= 0\,, & \delta b^*_{abc} &= \partial^d\partial_{[a}\rho_{bcd]}\,, \\ \partial^a g^*_{ab} - \frac{1}{2}\partial_b(\eta^{de}g^*_{de}) &= 0\,, & \delta g^*_{bc} &= \partial^a(\partial_a\omega_{bc} - 2\partial_{(b}\omega_{c)a}) + \partial_b\partial_c(\eta^{de}\omega_{de})\,, \\ \partial^a\chi^*_{a\beta} &= 0\,, & \delta\chi^*_{a\beta} &= (\gamma_{abc})_{\alpha\beta}\partial^b\xi^{c\alpha}\,. \end{aligned} \qquad (4.23)$$

As expected, the gauge invariances and equations of motion of the $d=11$ supergravity antifields are related to the equations of motion and gauge invariances of the $d=11$ supergravity fields of (4.17).

Using the wave function Ψ and the BRST operator $Q=\lambda^\alpha d_\alpha$, one can construct the spacetime action

$$\mathcal{S} = \int d^{11}x\langle\Psi Q\Psi\rangle \qquad (4.24)$$

where the norm $\langle\ \ \rangle$ is defined such that

$$\langle(\lambda^7\theta^9)\rangle = 1\,. \qquad (4.25)$$

Since $(\lambda^7\theta^9)$ is the scalar antighost state in (4.21) which cannot be written as $Q\Lambda$ for any Λ, the action of (4.24) is gauge invariant under $\delta\Psi = Q\Lambda$. Furthermore, the equations of motion from varying Ψ in (4.24) imply that $Q\Psi = 0$ for components in $Q\Psi$ involving up to nine θ's. Although the manifestly supersymmetric equations of motion require that $Q\Psi = 0$ for all components of $Q\Psi$, one can check that any component of $Q\Psi = 0$ with more than nine θ's is an auxiliary equation of motion which does not affect physical fields. So as in the super-Maxwell action of (2.25), removing these auxiliary equations of motion changes auxiliary fields to gauge fields, but does not change the physical content of the theory.

So the action of (4.24) reproduces the Batalin-Vilkovisky action for linearized $d = 11$ supergravity theory and, if Ψ is restricted to ghost number three, (2.25) reproduces the standard linearized $d = 11$ supergravity action. Note that the norm of (4.25) only involves integration over nine of the 32 θ^α variables and therefore resembles a harmonic superspace. Since there are 23 independent bosonic λ^α variables, one can interpret this integration over nine θ's as coming from a cancellation between the λ^α integration and the integration over 23 of the 32 θ^α variables.

5. Covariant quantization of the type-II superstring

In this section, the pure spinor description of the type-II superstring will be reviewed using language which will be convenient for generalization to the supermembrane.

5.1 Standard description of the type-II superstring

Using notation similar to that of the $N = 2$ $d = 10$ superparticle action of (3.1), the standard action for the type-II superstring can be written as

$$S = \int d\tau_0 d\tau_1 \left(P_m \Pi_0^m + B^{\rm flat}_{MN}\partial_{[0} Z^M \partial_{1]} Z^N + e^0 (P^m P_m + \Pi_1^m \Pi_{1m}) + e^1 P_m \Pi_1^m \right) \tag{5.1}$$

where

$$\Pi_0^m = \partial_0 x^m + \frac{i}{2}(\theta_L \gamma^m \partial_0 \theta_L + \theta_R \gamma^m \partial_0 \theta_R)\,, \qquad \Pi_1^m = \partial_1 x^m + \frac{i}{2}(\theta_L \gamma^m \partial_1 \theta_L + \theta_R \gamma^m \partial_1 \theta_R)\,, \tag{5.2}$$

e^0 and e^1 are Lagrange multipliers for the worldsheet reparameterization constraints, $B^{\rm flat}_{MN}$ is the flat value of the type-IIA two-form superfield, $Z^M = (x^m, \theta_L^\mu, \theta_R^{\hat\mu})$, and $M = (m, \mu, \hat\mu)$ is a $d = 10$ superspace coordinate. Note that after integrating out P^m, e^0 and e^1, the action of (5.1) reduces to the usual Nambu-Goto form of the GS superstring action.

Like the $N = 2$ $d = 10$ superparticle action of (3.1), the superstring action of (5.1) is invariant under global $N = 2$ $d = 10$ supersymmetry transformations and under "left-moving" and "right-moving" kappa transformations. To check kappa symmetry, note that under the local transformation

$$\begin{aligned} &\delta\theta_L^\mu = \xi_L^\mu\,, \qquad \delta\theta_R^{\hat\mu} = \xi_R^{\hat\mu}\,, \qquad \delta x^m = -\frac{i}{2}(\theta_L \gamma^m \xi_L + \theta_R \gamma^m \xi_R)\,, \\ &\delta P_m = -i(\xi_L \gamma_m \partial_1 \theta_L - \xi_R \gamma_m \partial_1 \theta_R)\,, \end{aligned} \tag{5.3}$$

the action of (5.1) transforms as

$$\delta S = i\int d^2\tau\,[(\xi_L\gamma^m\partial_R\theta_L)(P_m-\Pi_{1m})+(\xi_R\gamma^m\partial_L\theta_R)(P_m+\Pi_{1m})] \tag{5.4}$$

where $\partial_R=\partial_0+(e^1-2e^0)\partial_1$ and $\partial_L=\partial_0+(e^1+2e^0)\partial_1$

So if

$$\xi_L^\mu=(P_m-\Pi_{1m})(\gamma^m\kappa_L)^\mu \qquad \text{and} \qquad \xi_R^{\hat\mu}=(P_m+\Pi_{1m})(\gamma^m\kappa_R)^{\hat\mu} \tag{5.5}$$

for some $\kappa_{L\mu}$ and $\kappa_{R\hat\mu}$,

$$\delta S = i\int d^2\tau\left[(\kappa_{L\nu}\partial_R\theta_L^\nu)(P-\Pi_1)^2+(\kappa_{R\hat\nu}\partial_L\theta_R^{\hat\nu})(P+\Pi_1)^2\right], \tag{5.6}$$

which is cancelled by defining e^0 and e^1 to transform as

$$\delta e^0=-i\kappa_{L\nu}\partial_R\theta_L^\nu-i\kappa_{R\hat\nu}\partial_L\theta_R^{\hat\nu}\,,\qquad \delta e^1=2i\kappa_{L\nu}\partial_R\theta_L^\nu-2i\kappa_{R\hat\nu}\partial_L\theta_R^{\hat\nu}\,. \tag{5.7}$$

The canonical momenta to θ_L^μ and $\theta_R^{\hat\mu}$, which will be called $p_{L\mu}$ and $p_{R\hat\mu}$, satisfy

$$\begin{aligned} p_{L\mu} &= \frac{\partial L}{\partial\dot\theta_L^\mu}=\frac{i}{2}P^m(\gamma_m\theta_L)_\mu-B^{\rm flat}_{\mu N}\partial_1Z^N\,,\\ p_{R\hat\mu} &= \frac{\partial L}{\partial\dot\theta_R^{\hat\mu}}=\frac{i}{2}P^m(\gamma_m\theta_R)_{\hat\mu}-B^{\rm flat}_{\hat\mu N}\partial_1Z^N\,,\end{aligned}$$

so canonical quantization requires that physical states are annihilated by the 32 fermionic Dirac constraints defined by

$$d_{L\mu}=p_{L\mu}-\frac{i}{2}P_m(\gamma^m\theta_L)_\mu+B^{\rm flat}_{\mu N}\partial_1Z^N\,,\qquad d_{R\hat\mu}=p_{R\hat\mu}-\frac{i}{2}P_m(\gamma^m\theta_R)_{\hat\mu}+B^{\rm flat}_{\hat\mu N}\partial_1Z^N\,. \tag{5.8}$$

Using $\{p_{L\mu},\theta_L^\nu\}=-i\delta_\mu^\nu$, $\{p_{R\hat\mu},\theta_R^{\hat\nu}\}=-i\delta_{\hat\mu}^{\hat\nu}$, and the flat space value of $H^{\rm flat}_{MNP}=\partial_{[M}B^{\rm flat}_{NP]}$, one finds that these constraints satisfy the Poisson brackets

$$\{d_{L\mu},d_{L\nu}\}=-(P_m-\Pi_{1m})\gamma^m_{\mu\nu}\,,\qquad \{d_{R\hat\mu},d_{R\hat\nu}\}=-(P_m+\Pi_{1m})\gamma^m_{\hat\mu\hat\nu}\,,\qquad \{d_{L\mu},d_{R\hat\nu}\}=0\,. \tag{5.9}$$

And since $(P-\Pi_1)^2=0$ and $(P+\Pi_1)^2=0$ are also constraints, 16 of the 32 Dirac constraints are first-class and 16 are second-class. One can easily check that the 16 first-class Dirac constraints generate the kappa transformations of (5.3), however, there is no simple way to covariantly separate out the second-class constraints.

Although one cannot covariantly quantize the action of (5.1), one can classically couple the superstring to a type-II supergravity background using the action

$$\widehat{S}=\int d^2\tau\,[P_{\underline{m}}\Pi_0^{\underline{m}}+B_{MN}\partial_{[0}Z^M\partial_{1]}Z^N+e^0(P^{\underline{m}}P_{\underline{m}}+\Pi_1^{\underline{m}}\Pi_{1\underline{m}})+e^1P_{\underline{m}}\Pi_1^{\underline{m}}] \tag{5.10}$$

where $\Pi_0^{\underline{m}} = E_M^{\underline{m}}\partial_0Z^M$, $\Pi_1^{\underline{m}} = E_M^{\underline{m}}\partial_1Z^M$, $(E_{\underline{m}}^M, E_{\underline{\mu}}^M, E_{\underline{\hat\mu}}^M)$ is the super-vierbein, $(E_M^{\underline{m}}$,

$E^{\underline{\mu}}_M$,$E^{\underline{\hat\mu}}_M$) is the inverse super-vierbein, B_{MN} is the curved type-II two-form superfield, $M=[m,\mu,\hat\mu]$ denote curved vector and spinor indices, and the underlined indices $\underline{M}=[\underline{m},\underline{\mu},\underline{\hat\mu}]$ denote tangent-space vector and spinor indices[11].

This action is invariant under $N=2$ $d=10$ super-reparameterizations of the background as well as under the two-form gauge transformations $\delta B_{MN}=\partial_{[M}\Lambda_{N]}$. Under the local transformation

$$\delta Z^M = E^M_{\underline{\mu}}\xi^{\underline{\mu}}_L + E^M_{\underline{\hat\mu}}\xi^{\underline{\hat\mu}}_R\,,\qquad \delta P_{\underline{m}} = -i(\xi_L\gamma_{\underline{m}})_{\underline{\mu}}E^{\underline{\mu}}_M\partial_1 Z^M + i(\xi_R\gamma_{\underline{m}})_{\underline{\hat\mu}}E^{\underline{\hat\mu}}_M\partial_1 Z^M\,, \tag{5.11}$$

the action transforms as

$$\delta\widehat{S} = i\int d^2\tau\,\Big[(\xi_L\gamma^{\underline{m}})_{\underline{\mu}}E^{\underline{\mu}}_M\partial_R Z^M(P_{\underline{m}}-\Pi_{1\underline{m}}) + (\xi_R\gamma^{\underline{m}})_{\underline{\hat\mu}}E^{\underline{\hat\mu}}_M\partial_L Z^M(P_{\underline{m}}+\Pi_{1\underline{m}})\Big] \tag{5.12}$$

when the background superfields are on-shell. So the action is invariant under kappa symmetry if one defines

$$\delta e^0 = -i\kappa_{L\underline{\nu}}E^{\underline{\nu}}_M\partial_R Z^M - i\kappa_{R\underline{\hat\nu}}E^{\underline{\hat\nu}}_M\partial_L Z^M\,,\qquad \delta e^1 = 2i\kappa_{L\underline{\nu}}E^{\underline{\nu}}_M\partial_R Z^M - 2i\kappa_{R\underline{\hat\nu}}E^{\underline{\hat\nu}}_M\partial_L Z^M\,, \tag{5.13}$$

where

$$\xi^{\underline{\mu}}_L = (P_{\underline{m}}-\Pi_{1\underline{m}})(\gamma^{\underline{m}}\kappa_L)^{\underline{\mu}} \qquad\text{and}\qquad \xi^{\underline{\hat\mu}}_R = (P_{\underline{m}}+\Pi_{1\underline{m}})(\gamma^{\underline{m}}\kappa_R)^{\underline{\hat\mu}} \tag{5.14}$$

for some $\kappa_{L\mu}$ and $\kappa_{R\hat\mu}$.

5.2 Pure spinor description of the type-IIA superstring

Using notation similar to that of the pure spinor version of the $N=2$ $d=10$ superparticle, the pure spinor version of the type-II superstring action can be written as

$$\begin{aligned} S_{\text{pure}} = \int d\tau_0 d\tau_1\Big[&P_m\partial_0 x^m + p_{L\mu}\partial_0\theta^\mu_L + p_{R\hat\mu}\partial_0\theta^{\hat\mu}_R + w_{L\mu}\partial_0\lambda^\mu_L + w_{R\hat\mu}\partial_0\lambda^{\hat\mu}_R \\ &-\frac{1}{2}(P^m P_m + \partial_1 x^m\partial_1 x_m) + p_{L\mu}\partial_1\theta^\mu_L - p_{R\hat\mu}\partial_1\theta^{\hat\mu}_R + w_{L\mu}\partial_1\lambda^\mu_L - w_{R\hat\mu}\partial_1\lambda^{\hat\mu}_R + \\ &+ e_1\Big(P_m\partial_1 x^m + p_{L\mu}\partial_1\theta^\mu_L + p_{R\hat\mu}\partial_1\theta^{\hat\mu}_R + w_{L\mu}\partial_1\lambda^\mu_L + w_{R\hat\mu}\partial_1\lambda^{\hat\mu}_R\Big)\Big] \end{aligned} \tag{5.15}$$

where $p_{L\mu}$ and $p_{R\hat\mu}$ are now independent variables [24], λ^μ_L and $\lambda^{\hat\mu}_R$ are pure spinor ghost variables satisfying

$$\lambda_L\gamma^m\lambda_L = 0 \qquad\text{and}\qquad \lambda_R\gamma^m\lambda_R = 0 \qquad \text{for } m=1 \text{ to } 10\,, \tag{5.16}$$

and $w_{L\mu}$ and $w_{R\hat\mu}$ are defined up to the gauge transformations

$$\delta w_{L\mu} = (\gamma^m\lambda_L)_\mu\Lambda_{Lm}\,,\qquad \delta w_{R\hat\mu} = (\gamma^m\lambda_R)_{\hat\mu}\Lambda_{Rm}\,, \tag{5.17}$$

for arbitrary gauge parameters Λ_{Lm} and Λ_{Rm}. The action of (5.15) is quadratic in conformal gauge where e^1 is gauged to zero, but for later comparison with the supermembrane action, it will be useful to leave e^1 in the action and not fix reparameterization invariance

[11]To avoid confusion, the indices $a,b,c,\ldots$ and $\alpha,\beta,\gamma,\ldots$ will be reserved for $d=11$ indices.

of the τ_1 coordinate. Note, however, that like the pure spinor version of the superparticle actions, reparameterization invariance of the τ_0 coordinate has been fixed in (5.15). The action of (5.15) can be written in manifestly spacetime supersymmetric notation as

$$S_{\text{pure}} = \int d^2\tau \Big[\widetilde{P}_m \Pi_0^m + B^{\text{flat}}_{MN} \partial_{[0} Z^M \partial_{1]} Z^N + d_{L\mu}\partial_0\theta_L^\mu + d_{R\hat\mu}\partial_0\theta_R^{\hat\mu} + w_{L\mu}\partial_0\lambda_L^\mu + w_{R\hat\mu}\partial_0\lambda_R^{\hat\mu} - \\ - \frac{1}{2}(\widetilde{P}^m\widetilde{P}_m + \Pi_1^m\Pi_{1m}) + d_{L\mu}\partial_1\theta_L^\mu - d_{R\hat\mu}\partial_1\theta_R^{\hat\mu} + w_{L\mu}\partial_1\lambda_L^\mu - w_{R\hat\mu}\partial_1\lambda_R^{\hat\mu} + \\ + e_1(\widetilde{P}_m\Pi_1^m + d_{L\mu}\partial_1\theta_L^\mu + d_{R\hat\mu}\partial_1\theta_R^{\hat\mu} + w_{L\mu}\partial_1\lambda_L^\mu + w_{R\hat\mu}\partial_1\lambda_R^{\hat\mu}) \Big] \tag{5.18}$$

where Π^m, $d_{L\mu}$ and $d_{R\hat\mu}$ are defined as in (5.2) and (5.8), $\widetilde{P}_m = P_m - B^{\text{flat}}_{mN}\partial_1 Z^N$, and $Z^M = (x^m, \theta_L^\mu, \theta_R^{\hat\mu})$. Note that $\widetilde{P}_m$, $d_{L\mu}$ and $d_{R\hat\mu}$ are defined to be invariant under spacetime supersymmetry.

As in the $N = 2$ $d = 10$ superparticle, physical states of the type-II superstring are defined as states in the cohomology of the left and right-moving

$$Q_L = \lambda_L^\mu d_{L\mu} \qquad \text{and} \qquad Q_R = \lambda_R^{\hat\mu} d_{R\hat\mu}\,. \tag{5.19}$$

In other words, physical states Ψ will be defined by the equations of motion and gauge invariances

$$Q_L\Psi = Q_R\Psi = 0\,, \qquad \delta\Psi = Q_L\Lambda_L + Q_R\Lambda_R\,, \tag{5.20}$$

where the gauge parameters Λ_L and Λ_R are constrained to satisfy $Q_R\Lambda_L = Q_L\Lambda_R = 0$. Note that Q_L and Q_R are conserved using the equations of motion

$$\partial_R\lambda_L^\mu = \partial_R d_{L\mu} = 0\,, \qquad \partial_L\lambda_R^{\hat\mu} = \partial_L d_{R\hat\mu} = 0\,, \tag{5.21}$$

where $\partial_L = \partial_0 + (e^1 - 1)\partial_1$ and $\partial_R = \partial_0 + (e^1 + 1)\partial_1$. And using (5.9), one finds that $Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0$.

Since the superstring action of (5.15) reduces to the $N = 2$ $d = 10$ superparticle action when all worldsheet variables are independent of τ_1, the massless sector of the superstring spectrum consists of the type-II supergravity states found in subsection 3.3. Furthermore, it was shown in [19] that the massive states in the cohomology of Q_L and Q_R reproduce the standard superstring spectrum.

5.3 Coupling the superstring to type-II supergravity

As in subsection 2.4 for the $N = 1$ $d = 10$ superparticle in a super-Maxwell background, the easiest way to obtain the pure spinor version of the superstring action in a curved type-II supergravity background is to use the Oda-Tonin method of [28] and first compute the BRST variation of the standard superstring sigma model action of (5.10). Under the BRST transformation generated by $Q_L = \oint \lambda_L^\mu d_{L\mu}$ and $Q_R = \oint \lambda_R^{\hat\mu} d_{R\hat\mu}$ in a flat background,

$$Q_L\theta_L^\mu = \lambda_L^\mu\,, \quad Q_L x^m = \frac{i}{2}\lambda_L\gamma^m\theta_L\,, \quad Q_L d_{L\mu} = -i\Pi_L^m(\gamma_m\lambda_L)_\mu\,, \quad Q_L w_{L\mu} = d_{L\mu}\,, \\ Q_R\theta_R^{\hat\mu} = \lambda_R^{\hat\mu}\,, \quad Q_R x^m = \frac{i}{2}\lambda_R\gamma^m\theta_R\,, \quad Q_R d_{R\hat\mu} = -i\Pi_R^m(\gamma_m\lambda_R)_{\hat\mu}\,, \quad Q_R w_{R\hat\mu} = d_{R\hat\mu}\,, \tag{5.22}$$

where the equation of motion $\widetilde{P}^m = \Pi_0^m + e_1\Pi_1^m$ has been used, $\Pi_L^m = \Pi_0^m + (e_1-1)\Pi_1^m$, and $\Pi_R^m = \Pi_0^m + (e_1+1)\Pi_1^m$. In a curved background, these BRST transformations generalize to

$$\begin{aligned} \widehat{Q}_L Z^M &= E_{\underline{\mu}}^M \lambda_L^{\underline{\mu}}\,, & \widehat{Q}_L d_{L\underline{\mu}} &= -i\Pi_L^{\underline{m}}(\gamma_{\underline{m}}\lambda_L)_{\underline{\mu}}\,, & \widehat{Q}_L w_{L\underline{\mu}} &= d_{L\underline{\mu}}\,, \\ \widehat{Q}_R Z^M &= E_{\underline{\hat{\mu}}}^M \lambda_R^{\underline{\hat{\mu}}}\,, & \widehat{Q}_R d_{R\underline{\hat{\mu}}} &= -i\Pi_R^{\underline{m}}(\gamma_{\underline{m}}\lambda_R)_{\underline{\hat{\mu}}}\,, & \widehat{Q}_R w_{R\underline{\hat{\mu}}} &= d_{R\underline{\hat{\mu}}}\,, \end{aligned} \tag{5.23}$$

where $E_{\underline{\mu}}^M$ and $E_{\underline{\hat{\mu}}}^M$ are defined as in (5.10), $\Pi_L^{\underline{m}} = E_M^{\underline{m}}\partial_L Z^M$ and $\Pi_R^{\underline{m}} = E_M^{\underline{m}}\partial_R Z^M$.

After fixing the reparameterization gauge $e^0 = -1/2$ and using the equation of motion for $P_{\underline{m}}$, the standard superstring sigma model action of (5.10) transforms under (5.23) as

$$\widehat{Q}_L\widehat{S} = i\int d^2\tau(\lambda_L\gamma^{\underline{m}})_{\underline{\mu}}E_M^{\underline{\mu}}\partial_R Z^M\Pi_{L\underline{m}}\,, \qquad \widehat{Q}_R\widehat{S} = i\int d^2\tau(\lambda_R\gamma^{\underline{m}})_{\underline{\hat{\mu}}}E_M^{\underline{\hat{\mu}}}\partial_L Z^M\Pi_{R\underline{m}} \tag{5.24}$$

when the background superfields are on-shell. If $\lambda_L^{\underline{\mu}}$ and $\lambda_R^{\underline{\hat{\mu}}}$ were replaced by $\xi_L^{\underline{\mu}}$ and $\xi_R^{\underline{\hat{\mu}}}$ of (5.14), this would be a left and right-moving kappa transformation, which could be cancelled by shifting e^0 and e^1 as in (5.13). However, in the pure spinor formalism, the transformation of (5.24) will be cancelled by adding to the action the term

$$S' = \int d^2\tau\left[\widehat{Q}_L(w_{L\underline{\mu}}E_M^{\underline{\mu}}\partial_R Z^M) + \widehat{Q}_R(w_{R\underline{\hat{\mu}}}E_M^{\underline{\hat{\mu}}}\partial_L Z^M) - \widehat{Q}_L\widehat{Q}_R(w_{L\underline{\mu}}w_{R\underline{\hat{\nu}}}R^{\underline{\mu}\underline{\hat{\nu}}})\right] \tag{5.25}$$

where $R^{\underline{\mu}\underline{\hat{\nu}}}$ is a superfield whose lowest component is e^ϕ times the Ramond-Ramond field strength $F^{\underline{\mu}\underline{\hat{\nu}}}$. Using Bianchi identities and equations of motion, one can show that $R^{\underline{\mu}\underline{\hat{\nu}}}$ is related to the superspace torsion $T_{\underline{MN}}^{\underline{P}}$ by [21]

$$T_{\underline{\rho}\underline{m}}^{\underline{\hat{\nu}}} = i\gamma_{\underline{m}\underline{\rho}\underline{\mu}}R^{\underline{\mu}\underline{\hat{\nu}}}\,, \qquad T_{\underline{\hat{\rho}}\underline{m}}^{\underline{\mu}} = -i\gamma_{\underline{m}\underline{\hat{\rho}}\underline{\hat{\nu}}}R^{\underline{\mu}\underline{\hat{\nu}}}\,. \tag{5.26}$$

To see that $\widehat{Q}_L(\widehat{S}+S') = 0$, note that since $\widehat{Q}_L^2$ annihilates all variables except for $w_{L\underline{\mu}}$,

$$\begin{aligned} \widehat{Q}_L S' &= \int d^2\tau\left[(\widehat{Q}_L^2 w_{L\underline{\mu}})E_M^{\underline{\mu}}\partial_R Z^M - \widehat{Q}_R(w_{R\underline{\hat{\mu}}}\widehat{Q}_L(E_M^{\underline{\hat{\mu}}}\partial_L Z^M)) - \widehat{Q}_R((\widehat{Q}_L^2 w_{L\underline{\mu}})w_{R\underline{\hat{\nu}}}R^{\underline{\mu}\underline{\hat{\nu}}})\right] \\ &= \int d^2\tau\Big[-i\Pi_L^{\underline{m}}(\lambda_L\gamma_{\underline{m}})_{\underline{\mu}}E_M^{\underline{\mu}}\partial_R Z^M - \widehat{Q}_R(w_{R\underline{\hat{\mu}}}\lambda_L^{\underline{\nu}}T_{\underline{\nu}\underline{m}}^{\underline{\hat{\mu}}}\Pi_L^{\underline{m}}) - \\ &\qquad - \widehat{Q}_R(-i\Pi_L^{\underline{m}}(\lambda_L\gamma_{\underline{m}})_{\underline{\mu}}w_{R\underline{\hat{\nu}}}R^{\underline{\mu}\underline{\hat{\nu}}})\Big] \\ &= \int d^2\tau\left[-i\Pi_L^{\underline{m}}(\lambda_L\gamma_{\underline{m}})_{\underline{\mu}}E_M^{\underline{\mu}}\partial_R Z^M\right] = -\widehat{Q}_L\widehat{S} \end{aligned} \tag{5.27}$$

where the type-II on-shell torsion constraints have been used. Similarly, one can show that $\widehat{Q}_R(\widehat{S}+S') = 0$.

So the classically BRST-invariant superstring action in a curved type-II background is given by $\widehat{S}_{\text{pure}} = \widehat{S} + S'$, however, to preserve quantum BRST invariance [21], one also needs to add the Fradkin-Tseytlin term $\alpha'\int d^2\tau\Phi r$ to the superstring action where r is the worldsheet curvature and $\Phi(x,\theta_L,\theta_R)$ is a scalar superfield whose lowest component is the spacetime dilaton. Using the BRST transformation of (5.23) to explicitly compute S'

of (5.25), one finds that the pure spinor action in a curved background is

$$
\begin{aligned}
\widehat{S}_{\text{pure}} = \int d^2\tau \Big[& \frac{1}{2}\Pi_L^{\underline{m}}\Pi_{R\underline{m}} + B_{MN}\partial_{[0}Z^M\partial_{1]}Z^N + d_{L\underline{\mu}}E_M^{\underline{\mu}}\partial_R Z^M + \\
& + d_{R\underline{\hat{\mu}}}E_M^{\underline{\hat{\mu}}}\partial_L Z^M + w_{L\underline{\mu}}\partial_R\lambda_L^{\underline{\mu}} + w_{R\underline{\hat{\mu}}}\partial_L\lambda_R^{\underline{\hat{\mu}}} + \Omega_{M\underline{\mu}}^{\underline{\nu}}\partial_R Z^M\lambda_L^{\underline{\mu}}w_{L\underline{\nu}} + \\
& + \widehat{\Omega}_{M\underline{\hat{\mu}}}^{\underline{\hat{\nu}}}\partial_L Z^M\lambda_R^{\underline{\hat{\mu}}}w_{R\underline{\hat{\nu}}} + R^{\underline{\mu}\underline{\hat{\nu}}}d_{L\underline{\mu}}d_{R\underline{\hat{\nu}}} + C_{\underline{\rho}}^{\underline{\hat{\mu}}\underline{\nu}}d_{R\underline{\hat{\mu}}}\lambda_L^{\underline{\rho}}w_{L\underline{\nu}} + \\
& + \widehat{C}_{\underline{\hat{\rho}}}^{\underline{\mu}\underline{\hat{\nu}}}d_{L\underline{\mu}}\lambda_R^{\underline{\hat{\rho}}}w_{R\underline{\hat{\nu}}} + S_{\underline{\sigma}\underline{\hat{\rho}}}^{\underline{\mu}\underline{\hat{\nu}}}\lambda_L^{\underline{\sigma}}w_{L\underline{\mu}}\lambda_R^{\underline{\hat{\rho}}}w_{R\underline{\hat{\nu}}} + \alpha' r\Phi \Big]
\end{aligned}
\tag{5.28}
$$

where the explicit relations between the background superfields appearing in (5.28) are explained in [21].

By computing the linearized contribution of the background superfields to $\widehat{S}_{\text{pure}}$ of (5.28), one obtains the integrated form of the massless type-II superstring vertex operator $\int d^2\tau V$. Since $\widehat{S}_{\text{pure}}$ is BRST invariant, $Q_L\int d^2\tau V$ and $Q_R\int d^2\tau V$ must vanish up to worldsheet equations of motion when Q_L and Q_R generate the BRST transformations of (5.22) in a flat background.

Once one has the integrated BRST-invariant vertex operator associated with a physical state, there is a simple method for obtaining the unintegrated BRST-invariant vertex operator associated with this state. Since $(Q_L+Q_R)\int d^2\tau V = 0$, $(Q_L+Q_R)V = \partial_i W^i$ for some ghost number one state W^i where $i=0$ or 1. And since $(Q_L+Q_R)^2 V = 0$, $(Q_L+Q_R)W^i = \epsilon^{ij}\partial_j U$ for some ghost number two state U satisfying $(Q_L+Q_R)U = 0$. This ghost number two state U is defined to be the unintegrated closed superstring vertex operator associated with the physical state represented by V. Using this method, one finds that the unintegrated massless vertex operator U associated with the linearized contribution to $\widehat{S}_{\text{pure}}$ is the ghost number two $N=2$ $d=10$ superparticle wavefunction $\Psi(\lambda_L,\lambda_R,x,\theta_L,\theta_R) = \lambda_L^\mu\lambda_R^{\hat{\nu}}A_{\mu\hat{\nu}}(x,\theta_L,\theta_R)$ of (3.13).

Closed superstring massless N-point tree amplitudes can be computed in a manifestly super-Poincaré covariant manner by evaluating correlation functions of $N-3$ massless integrated vertex operators V with three massless unintegrated vertex operators U. The normalization for the worldsheet zero modes is the "left-right" product of the norm of (2.26), $\langle(\lambda_L^3\theta_L^5)(\lambda_R^3\theta_R^5)\rangle = 1$, which implies that the amplitudes are gauge invariant and supersymmetric when the external states are on-shell.

6. Covariant quantization of the supermembrane

In this section, the methods developed in the previous sections for the $d=11$ superparticle and type-II superstring are generalized to construct a BRST-invariant action for the supermembrane. Almost all of the intuition needed for constructing this action comes from the requirements that the action reduces to the $d=11$ superparticle and type-IIA superstring actions in the appropriate limits. That is, in the limit where all worldvolume variables are independent of coordinates τ_1 and τ_2, the supermembrane action must reduce to the $d=11$ superparticle action of section 4. And in the limit when $P_{11}=0$, $x^{11}=\tau_2$, and all other worldvolume variables are independent of τ_2, the action must reduce to the type-IIA superstring action of section 5.

6.1 Standard description of the supermembrane

Using notation similar to that of the $d = 11$ superparticle action of (4.1), the standard action for the supermembrane can be written as

$$S = \int d\tau_0 d\tau_1 d\tau_2 \left(P_c \Pi_0^c + B^{\rm flat}_{ABC} \partial_{[0} Z^A \partial_1 Z^B \partial_{2]} Z^C + e^0 (P^c P_c + \det(\Pi_I^c \Pi_{Jc})) + e^I P_c \Pi_I^c \right) \tag{6.1}$$

where $I, J = 1$ to 2, $\det(\Pi_I^c \Pi_{Jc}) = (\Pi_1^c \Pi_{1c})(\Pi_2^d \Pi_{2d}) - (\Pi_1^c \Pi_{2c})^2$, $\Pi_0^c = \partial_0 x^c + \frac{i}{2}(\theta \Gamma^c \partial_0 \theta)$, $\Pi_I^c = \partial_I x^c + \frac{i}{2}(\theta \Gamma^c \partial_I \theta)$, e^0 and e^I are Lagrange multipliers for the worldsheet reparameterization constraints, $B^{\rm flat}_{ABC}$ is the flat value of the $d = 11$ three-form superfield, $Z^A = (x^a, \theta^\alpha)$, and $A = (a, \alpha)$ is a $d = 11$ superspace coordinate. Note that after integrating out P^m, e^0 and e^I, the action of (6.1) reduces to the usual Nambu-Goto form of the supermembrane action [4].

Like the $d = 11$ superparticle action of (4.1), the supermembrane action of (6.1) is invariant under global $d = 11$ supersymmetry transformations and under kappa transformations. To check kappa symmetry, note that under the local transformation

$$\delta\theta^\alpha = \xi^\alpha\,, \qquad \delta x^c = -\frac{i}{2}(\theta\Gamma^c\xi)\,, \qquad \delta P_c = -i(\xi\Gamma_{cd}\partial_I\theta)\Pi_J^d \epsilon^{IJ}\,, \qquad \delta e^I = 2ie^0\epsilon^{IJ}\xi_\alpha\partial_J\theta^\alpha\,, \tag{6.2}$$

the action of (6.1) transforms as

$$\delta S = i \int d^3\tau \xi^\alpha \left(\Gamma^c_{\alpha\beta} P_c - \frac{1}{2}\Gamma^{cd}_{\alpha\beta} \Pi_{Ic}\Pi_{Jd}\epsilon^{IJ} \right) \nabla\theta^\beta \tag{6.3}$$

where

$$\nabla\theta^\beta = (\partial_0 + e^I\partial_I)\theta^\beta - 2e^0 \Gamma^\beta_{e\gamma}\partial_I\theta^\gamma \Pi_J^e \epsilon^{IJ}\,. \tag{6.4}$$

So if

$$\xi^\alpha = \left(\Gamma_c^{\alpha\beta} P^c + \frac{1}{2}\Gamma_{cd}^{\alpha\beta}\Pi_I^c\Pi_J^d\epsilon^{IJ} \right) \kappa_\beta \tag{6.5}$$

for some κ_β,

$$\delta S = i \int d^3\tau \left[\kappa_\beta \nabla\theta^\beta (P^d P_d + \det(\Pi_I^c\Pi_{Jc})) + 2(\kappa\Gamma^c\nabla\theta)\Pi_{Ic}(P^d\Pi_{Jd})\epsilon^{IJ} \right]. \tag{6.6}$$

So δS can be cancelled if e^0 and e^I are defined to transform as

$$\delta e^0 = -i\kappa_\beta\nabla\theta^\beta\,, \qquad \delta e^I = 2ie^0\epsilon^{IJ}\xi_\alpha\partial_J\theta^\alpha + 2i\epsilon^{IJ}(\kappa\Gamma^c\nabla\theta)\Pi_{Jc} \tag{6.7}$$

where the first term in δe^I comes from the transformation of (6.2).

The canonical momenta to θ^α, which will be called p_α, satisfies

$$p_\alpha = \frac{\partial L}{\partial\dot\theta^\alpha} = \frac{i}{2}P^c(\Gamma_c\theta)_\alpha - \frac{1}{2}B^{\rm flat}_{\alpha BC}\partial_I Z^B\partial_J Z^C\epsilon^{IJ}\,,$$

so canonical quantization requires that physical states are annihilated by the 32 fermionic Dirac constraints defined by

$$d_\alpha = p_\alpha - \frac{i}{2}P_c(\Gamma^c\theta)_\alpha + \frac{1}{2}B^{\rm flat}_{\alpha BC}\partial_I Z^B\partial_J Z^C\epsilon^{IJ}\,. \tag{6.8}$$

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Using $\{p_\alpha, \theta^\beta\} = -i\delta_\alpha^\beta$ and the flat space value of $H^{\rm flat}_{ABCD} = \partial_{[A}B^{\rm flat}_{BCD]}$, one finds that these constraints satisfy the Poisson brackets

$$\{d_\alpha, d_\beta\} = -P_c\Gamma^c_{\alpha\beta} + \frac{1}{2}\epsilon^{IJ}\Pi_{Ic}\Pi_{Jd}\Gamma^{cd}_{\alpha\beta}\,. \tag{6.9}$$

And since

$$\begin{aligned}\left(-P_c\Gamma^c_{\alpha\beta} + \frac{1}{2}\epsilon^{IJ}\Pi_{Ic}\Pi_{Jd}\Gamma^{cd}_{\alpha\beta}\right)\left(P^a\Gamma_a^{\beta\gamma} + \frac{1}{2}\epsilon^{KL}\Pi^a_K\Pi^b_L\Gamma^{\beta\gamma}_{ab}\right) &= \\ = -\delta^\gamma_\alpha(P_cP^c + \det(\Pi^c_I\Pi_{Jc})) - 2\epsilon^{IJ}(P^c\Pi_{Ic})\Pi_{Jd}\Gamma^{d\gamma}_\alpha&\end{aligned} \tag{6.10}$$

is proportional to the reparameterization constraints, 16 of the 32 Dirac constraints are first-class and 16 are second-class. One can easily check that the 16 first-class Dirac constraints generate the kappa transformations of (6.2), however, there is no simple way to covariantly separate out the second-class constraints.

Although one cannot covariantly quantize the action of (6.1), one can classically couple the supermembrane to a $d = 11$ supergravity background using the action

$$\widehat{S} = \int d^3\tau \left[P_{\underline{c}}\Pi^{\underline{c}}_0 + B_{ABC}\partial_{[0}Z^A\partial_1 Z^B\partial_{2]}Z^C + e^0(P^{\underline{c}}P_{\underline{c}} + \det(\Pi^{\underline{c}}_I\Pi_{J\underline{c}})) + e^I P_{\underline{c}}\Pi^{\underline{c}}_I\right] \tag{6.11}$$

where $\Pi^{\underline{c}}_0 = E^{\underline{c}}_A\partial_0 Z^A$, $\Pi^{\underline{c}}_I = E^{\underline{c}}_A\partial_I Z^A$, $[E^A_{\underline{c}}, E^A_{\underline{\alpha}}]$ is the super-vierbein, $[E^{\underline{c}}_A, E^{\underline{\alpha}}_A]$ is the inverse super-vierbein, B_{ABC} is the curved three-form superfield, $A = [a, \alpha]$ denote curved vector and spinor indices, and the underlined indices $\underline{A} = [\underline{c}, \underline{\alpha}]$ denote $d = 11$ tangent-space vector and spinor indices.

This action is invariant under $d = 11$ super-reparameterizations of the background as well as under the three-form gauge transformations $\delta B_{ABC} = \partial_{[A}\Lambda_{BC]}$. Under the local transformation

$$\delta Z^A = E^A_{\underline{\alpha}}\xi^{\underline{\alpha}}, \qquad \delta P_{\underline{c}} = -i(\xi\Gamma_{\underline{cd}})_{\underline{\alpha}}E^{\underline{\alpha}}_A\partial_I Z^A\Pi^{\underline{d}}_J\epsilon^{IJ}\,, \qquad \delta e^I = 2ie^0\epsilon^{IJ}\xi_{\underline{\alpha}}E^{\underline{\alpha}}_A\partial_J Z^A\,, \tag{6.12}$$

the action transforms as

$$\delta\widehat{S} = i\int d^3\tau\,\xi^{\underline{\alpha}}\left(\Gamma^{\underline{c}}_{\underline{\alpha\beta}}P_{\underline{c}} - \frac{1}{2}\Gamma^{\underline{cd}}_{\underline{\alpha\beta}}\Pi_{I\underline{c}}\Pi_{J\underline{d}}\epsilon^{IJ}\right)\nabla\Theta^{\underline{\beta}} \tag{6.13}$$

when the background is on-shell where

$$\nabla\Theta^{\underline{\beta}} = E^{\underline{\beta}}_A(\partial_0 + e^I\partial_I)Z^A - 2e^0\Gamma^{\underline{\beta}}_{\underline{a\gamma}}E^{\underline{\gamma}}_A\partial_I Z^A\Pi^{\underline{a}}_J\epsilon^{IJ}\,. \tag{6.14}$$

So the action is invariant under kappa symmetry if one defines

$$\delta e^0 = -i\kappa_{\underline{\beta}}\nabla\Theta^{\underline{\beta}}, \qquad \delta e^I = 2ie^0\epsilon^{IJ}\xi_{\underline{\alpha}}E^{\underline{\alpha}}_A\partial_J Z^A + 2i\epsilon^{IJ}(\kappa\Gamma^{\underline{c}}\nabla\Theta)\Pi_{J\underline{c}} \tag{6.15}$$

where

$$\xi^{\underline{\alpha}} = \left(\Gamma^{\underline{\alpha\beta}}_{\underline{c}}P^{\underline{c}} + \frac{1}{2}\Gamma^{\underline{\alpha\beta}}_{\underline{cd}}\Pi^{\underline{c}}_I\Pi^{\underline{d}}_J\epsilon^{IJ}\right)\kappa_{\underline{\beta}} \tag{6.16}$$

for some $\kappa_{\underline{\beta}}$.

6.2 Pure spinor description of the supermembrane

Using worldvolume variables which generalize the worldline variables of the pure spinor version of the $d = 11$ superparticle, the pure spinor version of the supermembrane action will be defined as

$$\begin{aligned} S_{\text{pure}} = \int d\tau_0 d\tau_1 d\tau_2 \Big[& \widetilde{P}_c \Pi_0^c + B^{\text{flat}}_{ABC} \partial_{[0} Z^A \partial_1 Z^B \partial_{2]} Z^C + d_\alpha \partial_0 \theta^\alpha + w_\alpha \partial_0 \lambda^\alpha - \\ & - \frac{1}{2} (\widetilde{P}^c \widetilde{P}_c + \det(\Pi_I^c \Pi_{Jc})) + (d\Gamma_c \partial_I \theta) \Pi_J^c \epsilon^{IJ} + (w\Gamma_c \partial_I \lambda) \Pi_J^c \epsilon^{IJ} - \\ & - i\epsilon^{IJ} (w\Gamma_c \partial_I \theta)(\lambda \Gamma^c \partial_J \theta) + i\epsilon^{IJ} (w_\alpha \partial_I \theta^\alpha)(\lambda_\beta \partial_J \theta^\beta) + \\ & + e^I (\widetilde{P}_c \Pi_I^c + d_\alpha \partial_I \theta^\alpha + w_\alpha \partial_I \lambda^\alpha) \Big] \end{aligned} \tag{6.17}$$

where $\widetilde{P}_c = P_c + \frac{1}{2} B^{\text{flat}}_{cAB} \partial_I Z^A \partial_J Z^B \epsilon^{IJ}$, d_α is defined as in (6.8), and $\widetilde{P}_c$ and d_α are defined to be invariant under supersymmetry transformations. One can easily check that this action reduces to the $d = 11$ superparticle action and type-IIA superstring action in the appropriate limits. Although the third line of (6.17) vanishes in these limits, the presence of the third line will be necessary for BRST invariance of the action. Note that the first and fourth lines of (6.17) simplify to $P_c \widehat{\partial}_0 x^c + p_\alpha \widehat{\partial}_0 \theta^\alpha + w_\alpha \widehat{\partial}_0 \lambda^\alpha$ where $\widehat{\partial}_0 = \partial_0 + e^I \partial_I$ when written in terms of the non-supersymmetric variables P_c and p_α. However, unlike the superstring action, the second line of (6.17) which comes from the supermembrane hamiltonian remains complicated when written in terms of P_c and p_α.

As in the $d = 11$ superparticle, the supermembrane action of (6.17) needs to be supplemented with the BRST-like constraint $Q = \lambda^\alpha d_\alpha$. Using the canonical commutation relations of (6.9), this constraint generates the BRST transformation

$$Q\theta^\alpha = \lambda^\alpha \,, \quad Qx^c = \frac{i}{2} \lambda \Gamma^c \theta \,, \quad Qd_\alpha = -i\widehat{\Pi}_0^c (\Gamma_c \lambda)_\alpha + \frac{i}{2} \epsilon^{IJ} \Pi_{Ic} \Pi_{Jd} (\Gamma^{cd} \lambda)_\alpha \,, \quad Qw_\alpha = d_\alpha \,, \tag{6.18}$$

where the equation of motion $\widetilde{P}^c = \widehat{\Pi}_0^c \equiv \Pi_0^c + e^I \Pi_I^c$ has been used. In addition, in order to allow the construction of a BRST-invariant action, it will be assumed that the Lagrange multipliers transform under a BRST transformation as

$$Qe^I = -i\epsilon^{IJ} \lambda_\alpha \partial_J \theta^\alpha \,. \tag{6.19}$$

The necessity of (6.19) can be seen from the kappa transformation of (6.2), and differs from the superstring kappa and BRST transformations of (5.3) and (5.22) where the Lagrange multipliers are invariant. This difference comes from the fact that supermembrane kappa transformations do not preserve the type-IIA superstring condition that $x^{11} = \tau_2$. So to restore the condition $x^{11} = \tau_2$ after performing a kappa or BRST transformation, one needs to perform a worldvolume reparameterization of τ_2 which transforms the Lagrange multipliers e^I.

A second important difference between the supermembrane BRST transformations of (6.18) is that $\lambda \Gamma^c \lambda = 0$ is not enough to guarantee that Q is nilpotent. Although

$Q^2\theta^\alpha = Q^2 x^c = 0$ when $\lambda\Gamma^c\lambda = 0$,

$$\begin{aligned} Q^2 d_\alpha &= (\lambda\Gamma^c\widehat{\partial}_0\theta)(\Gamma_c\lambda)_\alpha - \epsilon^{IJ}(\lambda\Gamma_c\partial_I\theta)\Pi_{Jd}(\Gamma^{cd}\lambda)_\alpha + \epsilon^{IJ}(\lambda_\beta\partial_I\theta^\beta)\Pi^c_J(\Gamma_c\lambda)_\alpha \\ &= -(\lambda\Gamma^c\Gamma^d\partial_I\theta)\Pi_{Jd}\epsilon^{IJ}(\Gamma_c\lambda)_\alpha - \epsilon^{IJ}(\lambda\Gamma_c\partial_I\theta)\Pi_{Jd}(\Gamma^{cd}\lambda)_\alpha + \epsilon^{IJ}(\lambda_\beta\partial_I\theta^\beta)\Pi^c_J(\Gamma_c\lambda)_\alpha \\ &= -(\lambda\Gamma^{cd}\partial_I\theta)\Pi_{Jd}\epsilon^{IJ}(\Gamma_c\lambda)_\alpha - \epsilon^{IJ}(\lambda\Gamma_c\partial_I\theta)\Pi_{Jd}(\Gamma^{cd}\lambda)_\alpha \\ &= \frac{1}{2}(\lambda\Gamma_{cd}\lambda)\Pi^c_I\epsilon^{IJ}(\Gamma^d\partial_J\theta)_\alpha\,, \end{aligned} \tag{6.20}$$

where the equation of motion $\nabla\theta^\alpha = 0$ has been used and

$$\nabla\theta^\alpha = (\partial_0 + e^I\partial_I)\theta^\alpha + (\Gamma^c\partial_I\theta)^\alpha\Pi_{Jc}\epsilon^{IJ}\,. \tag{6.21}$$

Furthermore, $Q^2 e^I = -i\epsilon^{IJ}\lambda_\alpha\partial_J\lambda^\alpha$.

For this reason, the pure spinor constraint $\lambda\Gamma^c\lambda = 0$ needs to be supplemented with the additional constraints $(\lambda\Gamma^{cd}\lambda)\Pi_{Jd} = 0$ and $\lambda_\alpha\partial_J\lambda^\alpha = 0$ in order that the BRST operator $Q = \lambda^\alpha d_\alpha$ is nilpotent. The fact that additional constraints are required for the supermembrane is not surprising since the supermembrane hamiltonian is not quadratic, implying that λ^α is not a free worldvolume variable. So the primary constraint $\lambda\Gamma^c\lambda = 0$ does not commute with the supermembrane hamiltonian and needs to be supplemented with secondary constraints using the standard Dirac procedure. To find these secondary constraints, note that under $\delta w_\alpha = (\Gamma^c\lambda)_\alpha\Lambda_c$,

$$\begin{aligned} \delta S_{\rm pure} &= \int d^2\tau\Lambda_c\left[\frac{1}{2}\widehat{\partial}_0(\lambda\Gamma^c\lambda) + (\lambda\Gamma^c\Gamma^d\partial_I\lambda)\Pi_{Jd}\epsilon^{IJ} - i\epsilon^{IJ}(\lambda\Gamma^{cd}\partial_I\theta)(\lambda\Gamma_d\partial_J\theta)\right] \\ &= \int d^2\tau\Lambda_c\Bigg[\frac{1}{2}\widehat{\partial}_0(\lambda\Gamma^c\lambda) + (\lambda_\alpha\partial_I\lambda^\alpha)\Pi^c_J\epsilon^{IJ} + \frac{1}{2}\partial_I[(\lambda\Gamma^{cd}\lambda)\Pi_{Jd}\epsilon^{IJ}] + \\ &\qquad + \frac{i}{4}(\lambda\Gamma^d\lambda)(\partial_I\theta\Gamma^{cd}\partial_J\theta)\epsilon^{IJ}\Bigg]\,. \end{aligned} \tag{6.22}$$

So $(\lambda\Gamma^{cd}\lambda)\Pi_{Jd} = 0$ and $\lambda_\alpha\partial_J\lambda^\alpha = 0$ are an appropriate choice of secondary constraints.

Note that the secondary constraints $(\lambda\Gamma^{cd}\lambda)\Pi_{Jd} = 0$ and $\lambda_\alpha\partial_J\lambda^\alpha = 0$ also do not commute with the hamiltonian, and therefore lead to further secondary constraints. However, one can easily check that the complete set of primary and secondary constraints generated in this manner is first-class. Furthermore, for checking nilpotence of Q and BRST invariance of the supermembrane action, only

$$\lambda\Gamma^c\lambda = 0, \quad (\lambda\Gamma^{cd}\lambda)\Pi_{Jc} = 0, \quad \lambda_\alpha\partial_J\lambda^\alpha = 0 \tag{6.23}$$

are required.

The most direct way to check BRST invariance of the supermembrane action is to show that (6.17) is invariant under (6.18) and (6.19) up to the constraints of (6.23). However, a more elegant way to show BRST invariance is to use the Oda-Tonin method of [28] and write the action of (6.17) as

$$S_{\rm pure} = S + \int d^3\tau\ Q[w_\alpha\nabla\theta^\alpha] \tag{6.24}$$

where $\nabla\theta^\alpha$ is defined in (6.21) and S is the standard supermembrane action of (6.1) in the reparameterization gauge $e^0 = -1/2$. Using (6.3) and replacing ξ^α with λ^α, one finds that

$$\begin{aligned} QS_{\text{pure}} &= QS + \int d^3\tau\, Q^2[w_\alpha \nabla\theta^\alpha] \\ &= i\int d^3\tau \left[\lambda^\alpha\left(\Gamma^c_{\alpha\beta}P_c - \frac{1}{2}\Gamma^{cd}_{\alpha\beta}\Pi_{Ic}\Pi_{Jd}\epsilon^{IJ}\right)\nabla\theta^\beta + Q^2(w_\alpha)\nabla\theta^\alpha\right] = 0\,, \end{aligned} \tag{6.25}$$

so S_{pure} is BRST invariant.

6.3 Coupling the supermembrane to a $d = 11$ supergravity background

By starting with the standard supermembrane action in a curved background and using the background-dependent version of the BRST transformations, one can also use the Oda-Tonin method to construct the BRST-invariant version of the supermembrane action in a curved background. In a curved background, the BRST transformations of (6.18) and (6.19) are generalized to

$$\begin{aligned} \widehat{Q}Z^B &= E^B_{\underline{\alpha}}\lambda^{\underline{\alpha}}\,, \qquad & \widehat{Q}d_{\underline{\alpha}} &= -i\widehat{\Pi}^{\underline{c}}_0(\Gamma_{\underline{c}}\lambda)_{\underline{\alpha}} + \tfrac{i}{2}\epsilon^{IJ}\Pi_{I\underline{c}}\Pi_{J\underline{d}}(\Gamma^{\underline{cd}}\lambda)_{\underline{\alpha}}\,, \\ \widehat{Q}w_{\underline{\alpha}} &= d_{\underline{\alpha}}\,, & \widehat{Q}e^I &= -i\epsilon^{IJ}\lambda_{\underline{\alpha}}E^{\underline{\alpha}}_A\partial_J Z^A\,. \end{aligned} \tag{6.26}$$

For $\widehat{Q}$ to be nilpotent in a curved background, one needs to impose the constraints

$$\lambda\Gamma^{\underline{c}}\lambda = 0, \quad (\lambda\Gamma^{\underline{cd}}\lambda)\Pi_{J\underline{c}} = 0\,, \qquad \lambda_{\underline{\alpha}}\nabla_J\lambda^{\underline{\alpha}} = 0\,, \tag{6.27}$$

where

$$\nabla_J\lambda^{\underline{\alpha}} \equiv \partial_J\lambda^{\underline{\alpha}} + \Omega^{\underline{cd}}_B(\Gamma_{\underline{cd}}\lambda)^{\underline{\alpha}}\partial_J Z^B + \lambda^{\underline{\beta}}T^{\underline{\alpha}}_{\underline{\beta c}}\Pi^{\underline{c}}_J\,, \tag{6.28}$$

$T^{\underline{\alpha}}_{\underline{\beta c}}$ is a dimension-one component of the superspace torsion which is related on-shell to the four-form field strength $H_{\underline{abcd}}$, and $\Omega^{\underline{cd}}_B$ is the superspace spin connection. Note that the third constraint of (6.27) is obtained from requiring that $\widehat{Q}^2e^I = 0$.

The resulting BRST-invariant action in a curved background is

$$\widehat{S}_{\text{pure}} = \widehat{S} + \int d^3\tau\widehat{Q}[w_{\underline{\alpha}}\nabla\Theta^{\underline{\alpha}}] \tag{6.29}$$

where

$$\nabla\Theta^{\underline{\beta}} = E^{\underline{\beta}}_A\widehat{\partial}_0 Z^A + \Gamma^{\underline{\beta}}_{\underline{a\gamma}}E^{\underline{\gamma}}_A\partial_I Z^A\Pi^{\underline{a}}_J\epsilon^{IJ} \tag{6.30}$$

and $\widehat{S}$ is defined in (6.11) after gauge-fixing $e^0 = -1/2$ and setting $P^{\underline{c}} = \widehat{\Pi}^{\underline{c}}_0$. Using the background-dependent BRST transformation of (6.26), one finds

$$\begin{aligned} \widehat{S}_{\text{pure}} = \int d^3\tau \Big[&\frac{1}{2}\widehat{\Pi}^{\underline{c}}_0\widehat{\Pi}_{0\underline{c}} + B_{ABC}\partial_{[0}Z^A\partial_1 Z^B\partial_{2]}Z^C - \frac{1}{2}\det(\Pi^{\underline{c}}_I\Pi_{J\underline{c}}) + d_{\underline{\alpha}}\nabla\Theta^{\underline{\alpha}} + \\ &+ w_{\underline{\alpha}}(\widehat{\nabla}_0\lambda^{\underline{\alpha}} + \Gamma^{\underline{\alpha}}_{\underline{c\beta}}(\nabla_I\lambda^{\underline{\beta}})\Pi^{\underline{c}}_J\epsilon^{IJ}) - i\epsilon^{IJ}(w_{\underline{\alpha}}\Gamma^{\underline{\alpha}}_{\underline{c\beta}}E^{\underline{\beta}}_B\partial_I Z^B)(\lambda_{\underline{\gamma}}\Gamma^{\underline{c\gamma}}_{\underline{\delta}}E^{\underline{\delta}}_A\partial_J Z^A) + \\ &+ i\epsilon^{IJ}(w_{\underline{\alpha}}E^{\underline{\alpha}}_B\partial_I Z^B)(\lambda_{\underline{\gamma}}E^{\underline{\gamma}}_A\partial_J Z^A)\Big] \end{aligned} \tag{6.31}$$

where $\nabla_0\lambda^{\underline{\beta}}$ and $\nabla_I\lambda^{\underline{\beta}}$ are defined as in (6.28).

At first sight, it might seem surprising that the pure spinor version of the supermembrane action in a curved background does not reduce to the type-IIA superstring sigma model action of (5.28) after double-dimensional reduction. Although both actions are BRST invariant, the $\lambda\Gamma^{11}\lambda = 0$ constraint in the supermembrane formalism implies that the curved background fields couple differently in the two actions. For example, there is no analog of the Fradkin-Tseytlin term $\alpha' \int d^2\tau \Phi r$ or the $\int d^2\tau R^{\underline{\mu}\underline{\hat{\nu}}} d_{L\underline{\mu}} d_{R\underline{\hat{\nu}}}$ term in the supermembrane action. However, since the type-IIA superstring sigma model is only valid for perturbative string theory, these two actions are only guaranteed to agree in the limit when the string coupling constant goes to zero. In fact, it is clear that the Fradkin-Tseytlin term $\alpha' \int d^2\tau \Phi r$ is not possible in the supermembrane action since there are no scalars which can play the role of the dilaton. Also, the term $\int d^2\tau R^{\underline{\mu}\underline{\hat{\nu}}} d_{L\underline{\mu}} d_{R\underline{\hat{\nu}}}$ vanishes when the string coupling constant goes to zero since $R^{\underline{\mu}\underline{\hat{\nu}}}$ is proportional to $e^{\phi} F^{\underline{\mu}\underline{\hat{\nu}}}$ where $F^{\underline{\mu}\underline{\hat{\nu}}}$ is the Ramond-Ramond field strength. As will be discussed in the following section, in order to relate the supermembrane with the type-IIA superstring at non-zero string coupling constant, one has to compute supermembrane scattering amplitudes.

7. Supermembrane scattering amplitudes

In this section, a prescription is given for computing scattering amplitudes using the supermembrane action. These amplitudes might be useful for studying non-perturbative properties of the type-IIA superstring.

7.1 Supermembrane massless vertex operators

In order to compute scattering amplitudes using the supermembrane action, one first needs to define BRST-invariant integrated and unintegrated massless supermembrane vertex operators. As in the superstring sigma model action of (5.28), the massless integrated supermembrane vertex operator $\int d^3\tau V$ can be defined as the linear contribution of the background superfields to $\widehat{S}_{\text{pure}}$ of (6.31). So V has the form

$$\begin{aligned} V = {} & A_{BC}\widehat{\partial}_0 Z^B \widehat{\partial}_0 Z^C + A_{BCD}\widehat{\partial}_0 Z^B \partial_I Z^C \partial_J Z^D \epsilon^{IJ} + \\ & + A_{BCDE}\partial_I Z^B \partial_J Z^C \partial_K Z^D \partial_L Z^E \epsilon^{IJ}\epsilon^{KL} + \\ & + (C^{\alpha}_{B}\widehat{\partial}_0 Z^B + C^{\alpha}_{BC}\partial_I Z^B \partial_J Z^C \epsilon^{IJ}) d_\alpha + \\ & + (\Omega^{\alpha}_{\beta C}\widehat{\partial}_0 Z^C + \Omega^{\alpha}_{\beta CD}\partial_I Z^C \partial_J Z^D \epsilon^{IJ}) w_\alpha \lambda^\beta + Y^{\alpha}_{\beta B}\partial_I Z^B w_\alpha \partial_J \lambda^\beta \epsilon^{IJ} \end{aligned} \tag{7.1}$$

where the relations between the various superfields in (7.1) are determined by the BRST invariance condition that $\int d^3\tau QV = 0$. Note that since the action $\widehat{S}_{\text{pure}}$ is invariant under the background-dependent BRST transformations of (6.26), $\int d^3\tau V$ is invariant up to equations of motion under the flat BRST transformations of (6.18) and (6.19).

To obtain the massless unintegrated supermembrane vertex operator, one can use the supermembrane version of the method used in subsection 5.3 for the superstring. Since $\int d^3\tau QV = 0$, $QV = \partial_i W^i$ where W^i are ghost number one states for $i = 0$ to 2. And since $Q^2 V = 0$, $QW^i = \epsilon^{ijk}\partial_j Y_k$ where Y_k are ghost number two states. Finally, since $Q^2 W^i = 0$, $QY_k = \partial_k U$ where U is a ghost number three BRST-invariant state which will be identified with the unintegrated supermembrane vertex operator. Using this method for the integrated massless vertex operator of (7.1), one finds that U is the ghost number three $d = 11$ superparticle wavefunction $\Psi(\lambda, x, \theta) = \lambda^\alpha \lambda^\beta \lambda^\gamma A_{\alpha\beta\gamma}(x, \theta)$ of (4.15). So one sees that $d = 11$ supergravity fields carry ghost number three since they couple to the three-dimensional supermembrane worldvolume, while $d = 10$ super-Maxwell and type-II supergravity fields carry ghost number one and two since they couple respectively to the one-dimensional superparticle worldline and two-dimensional superstring worldsheet.

7.2 Supermembrane scattering amplitude prescription

To compute supermembrane scattering amplitudes using these integrated and unintegrated massless vertex operators, one naively should evaluate correlation functions of these vertex operators on a supermembrane worldvolume. However, since there is no $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{C})$ of the supermembrane worldvolume, it is not obvious how many vertex operators should be unintegrated and how many should be integrated. Furthermore, since the supermembrane action is not conformally invariant, it is not clear what type of worldvolumes should be included in the correlation function.

One natural requirement is that the worldvolume zero modes in the correlation function should be normalized using the $d = 11$ superparticle norm $\langle(\lambda^7 \theta^9)\rangle$, where the spinor index contractions of λ^α and θ^β in $(\lambda^7 \theta^9)$ are those of (4.21). Since the state $(\lambda^7 \theta^9)$ is in the cohomology of Q and cannot be written as the supersymmetric variation of any state in the cohomology of Q, use of this zero mode normalization guarantees that the scattering amplitudes are gauge invariant and supersymmetric when the external states are on-shell.

The fact that $(\lambda^7 \gamma^9)$ carries ghost number seven implies that scattering amplitudes must involve more vertex operators than just the integrated vertex operator V of ghost number zero and the unintegrated vertex operator U of ghost number three. It will now be argued that a correct prescription for an N-point supermembrane scattering amplitude is to use $N - 2$ integrated vertex operators V of ghost number zero, one unintegrated vertex operator U of ghost number three, and one unintegrated vertex operator U^* of ghost number four. For massless external states, the ghost number three unintegrated vertex operator is the $d = 11$ superparticle wavefunction $U = \lambda^\alpha \lambda^\beta \lambda^\gamma A_{\alpha\beta\gamma}(x, \theta)$ for the linearized supergravity fields, while the ghost number four unintegrated vertex operator is the $d = 11$ superparticle wavefunction $U^* = \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta A^*_{\alpha\beta\gamma\delta}(x, \theta)$ for the linearized supergravity antifields.

Although this amplitude prescription may sound unusual, it will be shown below that it can also be used for N-point open and closed string tree amplitudes. The prescription in some sense resembles the old operator formalism for computing scattering amplitudes where two external strings are treated as initial and final states, and the remaining $N - 2$ external strings are treated as vertices. Using this interpretation, the open or closed string worldsheet is an infinitely long strip or cylinder of zero curvature, and the vertices represent infinitesimally short strings. So when used for supermembrane scattering amplitudes, this prescription only involves supermembrane worldvolumes of zero curvature. Note that unlike superstring amplitudes, one does not expect to expand over worldvolumes of different genus for supermembrane amplitudes since there is no scalar spacetime field whose expectation value could play the role of a dimensionless coupling constant.

In open string theory, the analogous prescription involves $N-2$ integrated vertex operators, one ghost number one unintegrated vertex operator U for the string field, and one ghost number two integrated vertex operator U^* for the string antifield. In bosonic open string theory, these vertex operators can be chosen in the Siegel gauge to satisfy $U = cV$ and $U^* = c\partial cV$ where V is a dimension one primary field which is independent of the (b, c) ghosts. Computing the correlation function

$$\mathcal{A} = \left\langle U_1(z_1) U_2^*(z_2) \int dz_3 V_3(z_3) \cdots \int dz_N V_N(z_N) \right\rangle , \tag{7.2}$$

one obtains

$$\mathcal{A} = (z_1 - z_2)^2 \left\langle V_1(z_1) V_2(z_2) \int dz_3 V_3(z_3) \cdots \int dz_N V_N(z_N) \right\rangle . \tag{7.3}$$

Since (7.3) is invariant under the SL$(2,\mathbb{R})$ transformation $z_r \to (az_r + b)/(cz_r + d)$, one can fix (z_1, z_2, z_3) so that the integral over $\int dz_3$ gives a trivial constant infinite factor which is independent of the external vertex operators. After dividing by this infinite constant factor, one recovers the standard open string N-point tree amplitude expression

$$\mathcal{A} = (z_1 - z_2)(z_3 - z_1)(z_3 - z_2) \left\langle V_1(z_1) V_2(z_2) V_3(z_3) \int dz_4 V_4(z_4) \cdots \int dz_N V_N(z_N) \right\rangle . \tag{7.4}$$

In closed string theory, the analogous prescription for N-point tree amplitudes involves $N-2$ integrated vertex operators, one ghost number two unintegrated vertex operator U for the string field, and one ghost number four unintegrated vertex operator U^* for the string antifield. In bosonic closed string theory in Siegel gauge, these unintegrated vertex operators are $U = c_L c_R V$ and $U^* = c_L(\partial_L c_L) c_R(\partial_R c_R) V$ where V is a dimension $(1,1)$ primary field which is independent of the ghosts. As in open string amplitudes, SL$(2,\mathbb{C})$ invariance of the amplitude

$$\mathcal{A} = \left\langle U_1(z_1) U_2^*(z_2) \int d^2 z_3 V_3(z_3) \cdots \int d^2 z_N V_N(z_N) \right\rangle \tag{7.5}$$

implies that the $\int d^2 z_3$ integral provides a trivial constant infinite factor. After dividing out this infinite factor, one recovers the standard expression for the closed string N-point tree amplitude.

Using the pure spinor version of open superstring field theory with massless external states, the unintegrated ghost number one vertex operator is $U = \lambda^\mu A_\mu(x, \theta)$ and the ghost number two vertex operator is $U^* = \lambda^\mu \lambda^\nu A^*_{\mu\nu}(x, \theta)$ where A_μ depends on the super-Maxwell photon a_m and photino χ^μ satisfying (2.21) and $A^*_{\mu\nu}$ depends on the super-Maxwell antiphoton a^*_m and antiphotino χ^*_μ satisfying (2.24). The analog of Siegel gauge for these fields and antifields is

$$a_m = a^*_m , \qquad \chi^\mu = \gamma_m^{\mu\nu} \partial^m \chi^*_\nu \tag{7.6}$$

using the gauge-fixing conditions

$$\partial^m a_m = 0 , \qquad \partial_n \partial^n a^*_m = \partial^n \partial_n \chi^*_\nu = 0 . \tag{7.7}$$

Note that the gauge-fixing conditions on the fields are the antifield equations of motion, while the gauge-fixing conditions on the antifields is that they are annihilated by $\partial^n\partial_n$. Also note that the identification of (7.6) is consistent with the equations of motion of (2.21) and (2.24) in this gauge. So after dividing out the constant factor and using (7.6) to map antifields into fields, open superstring massless tree amplitudes can be computed using this prescription.

Using the pure spinor version of closed superstring field theory with massless external states, the unintegrated ghost number two vertex operator is $U = \lambda_L^\mu \lambda_R^{\hat\nu} A_{\mu\hat\nu}(x,\theta_L,\theta_R)$ and the ghost number four vertex operator is $U^* = \lambda_L^\mu \lambda_L^\nu \lambda_R^{\hat\rho}\lambda_R^{\hat\sigma} A^*_{\mu\nu\hat\rho\hat\sigma}(x,\theta_L,\theta_R)$ where $A_{\mu\hat\nu}$ depends on the type-II supergravity fields $[a_{mn}, \chi_{Lm}^{\hat\mu}, \chi_{Rm}^{\mu}, F^{\mu\hat\nu}]$ satisfying (3.18) and (3.19), and $A^*_{\mu\nu\hat\rho\hat\sigma}$ depends on their antifields $[a^*_{mn}, \chi^*_{Lm\hat\mu}, \chi^*_{Rm\mu}, F^*_{\mu\hat\nu}]$. The analog of Siegel gauge for these type-II supergravity fields and antifields is

$$a_{mn} = a^*_{mn}\,, \quad \chi^{\hat\mu}_{Lm} = \gamma_n^{\hat\mu\hat\nu}\partial^n \chi^*_{Lm\hat\nu}\,, \quad \chi^{\mu}_{Rm} = \gamma_n^{\mu\nu}\partial^n \chi^*_{Rm\nu}\,, \quad F^{\mu\hat\nu} = \gamma_m^{\mu\rho}\gamma_n^{\hat\nu\hat\sigma}\partial^m\partial^n F^*_{\rho\hat\sigma}\,, \tag{7.8}$$

using the gauge-fixing conditions

$$\begin{aligned} \partial^m a_{mn} &= \partial^n a_{mn} = \partial^m \chi^{\hat\mu}_{Lm} = \partial^m \chi^{\mu}_{Rm} = 0\,, \\ \partial_n\partial^n a^*_{mn} &= \partial^n\partial_n \chi^*_{Lm\hat\nu} = \partial^n\partial_n \chi^*_{Rm\nu} = \partial^n\partial_n F^*_{\mu\hat\nu} = 0\,. \end{aligned} \tag{7.9}$$

The relations of (7.8) and (7.9) can be understood as the "left-right" product of the relations of (7.6) and (7.7). So after dividing out the constant factor and using (7.8) to map antifields into fields, open superstring massless tree amplitudes can be computed using this prescription.

Finally, using the pure spinor version of supermembrane theory with massless external states, the unintegrated ghost number three vertex operator is $U = \lambda^\alpha\lambda^\beta\lambda^\gamma A_{\alpha\beta\gamma}(x,\theta)$ and the ghost number four vertex operator is $U^* = \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta A^*_{\alpha\beta\gamma\delta}(x,\theta)$ where $A_{\alpha\beta\gamma}$ depends on the $d=11$ supergravity fields $[g_{bc}, b_{bcd}, \chi_b^\alpha]$ satisfying (4.17), and $A^*_{\alpha\beta\gamma\delta}$ depends on their antifields $[g^*_{bc}, b^*_{bcd}, \chi^*_{b\alpha}]$ satisfying (4.23). The analog of Siegel gauge for these $d=11$ supergravity fields and antifields is

$$g_{bc} = g^*_{bc}\,, \qquad b_{bcd} = b^*_{bcd}\,, \qquad \chi^\alpha_b = \Gamma_c^{\alpha\beta}\partial^c \chi^*_{b\beta} + \frac{1}{9}(\Gamma_b\Gamma^c\Gamma^d)^{\alpha\beta}\partial_c \chi^*_{d\beta} \tag{7.10}$$

using the gauge-fixing conditions

$$\partial^b g_{bc} - \frac{1}{2}\partial_c(\eta^{de}g_{de}) = \partial^b b_{bcd} = \partial^b \chi^\alpha_b = 0\,, \qquad \partial^b\partial_b g^*_{cd} = \partial^b\partial_b b^*_{cde} = \partial^b\partial_b \chi^*_{c\alpha} = 0\,. \tag{7.11}$$

The identification for the gravitino and its antifield in (7.10) can be obtained by requiring consistency of the gravitino equation of motion with the gauge-fixing conditions of (7.11). So as in the open and closed superstring, it should be possible to use the map of (7.10) to obtain $d=11$ supergravity amplitudes from the supermembrane amplitude prescription.

7.3 M-theory conjecture for supermembrane amplitudes

Having shown in the previous subsection that the prescription for supermembrane scattering amplitudes has an analog for the superstring, it is natural to ask if there is some relation between the supermembrane and superstring scattering amplitudes. It will now be conjectured that after compactification of x^{11} on a circle of radius R_{11}, the supermembrane massless scattering amplitude coincides with the non-perturbative type-IIA superstring massless scattering amplitude with the string coupling constant equal to $(R_{11})^{3/2}$. This conjecture is based of course on the M-theory conjecture of [2].

Since the supermembrane action of (6.17) is not quadratic, it will not be possible to obtain exact expressions for correlation functions of supermembrane vertex operators as was done for correlation functions of superstring vertex operators. However, since the supermembrane action reduces in the infinite tension limit to the $d=11$ superparticle action of (4.7) which is quadratic, it might be possible to compute supermembrane scattering amplitudes as a perturbative expansion in the inverse of the membrane tension. Hopefully, the non-renormalizability of the supermembrane action will not be an insurmountable obstacle in performing this perturbative expansion. Since the expansion in the membrane tension is manifestly $d=11$ super-Poincaré covariant, even the lowest order terms in the expansion will contain information about type-IIA superstring amplitudes that is non-perturbative in the string coupling constant.

The first step in studying this M-theory conjecture is to get a better understanding of the relation beween the supermembrane massless vertex operators and the type-IIA superstring massless vertex operators. Although the supermembrane action in a flat background reduces to the type-IIA superstring action in a flat background after double-dimensional reduction of x^{11}, the supermembrane action in a curved background does not reduce to the type-IIA superstring action in a curved background. Since the integrated version of the massless vertex operator comes from the linearized contribution of the curved background, this means that the integrated version of the supermembrane massless vertex operator does not reduce to the integrated version of the type-IIA massless vertex operator. For example, there is no term quadratic in d_α in the massless integrated supermembrane vertex operator of (7.1) which reduces to the $d_{L\mu}d_{R\hat{\nu}}R^{\mu\hat{\nu}}$ term in the massless integrated type-IIA superstring vertex operator. As discussed earlier, this difference comes from the $\lambda\Gamma^{11}\lambda=0$ constraint in the supermembrane formalism which is not present in the type-IIA superstring formalism.

Also, the unintegrated supermembrane vertex operators U_{membrane} and U^*_{membrane} of ghost number three and four are different from the unintegrated type-IIA superstring vertex operators U_{string} and U^*_{string} of ghost number two and four. However, in this case, there is a simple way to relate the supermembrane and superstring vertex operators as will now be shown.

Although $QU_{\text{membrane}}=0$ when $\lambda\Gamma^c\lambda=0$ for $c=1$ to 11, it is not necessarily zero if $\lambda\Gamma^m\lambda=0$ for $m=1$ to 10 but $\lambda\Gamma^{11}\lambda$ is non-zero. In this case, $QU_{\text{membrane}}=(\lambda\Gamma^{11}\lambda)Y$ where Y is some ghost number two state annihilated by Q. Note that because of (4.13), Y is defined up to terms proportional to $\lambda\Gamma^{m\ 11}\lambda=\lambda_L\gamma^m\lambda_L-\lambda_R\gamma^m\lambda_R$. Since $\lambda_L\gamma^m\lambda_L-\lambda_R\gamma^m\lambda_R=0$ using the pure spinor version of the type-IIA superstring, one can identify Y with U_{string}. In other words, after double-dimensional reduction,

$$QU_{\text{membrane}}=(\lambda\Gamma^{11}\lambda)U_{\text{string}}\,. \tag{7.12}$$

To relate U^*_{string} and U^*_{membrane}, first consider $(Q_L + Q_R)U^*_{\text{string}}$ when $\lambda_L\gamma^m\lambda_L + \lambda_R\gamma^m\lambda_R = 0$ but $\lambda_L\gamma^m\lambda_L - \lambda_R\gamma^m\lambda_R$ is non-zero. In this case,

$$(Q_L + Q_R)U^*_{\text{string}} = (\lambda_L\gamma^m\lambda_L - \lambda_R\gamma^m\lambda_R)S_m \tag{7.13}$$

where S_m is some ghost number three vector state which, because of (4.13), is defined up to terms proportional to $\lambda\Gamma^{11}\lambda = \lambda_L^\mu\lambda_{R\mu}$. Furthermore, $(Q_L + Q_R)^2U^*_{\text{string}} = 0$ and $\lambda_L\gamma^m\lambda_L + \lambda_R\gamma^m\lambda_R = 0$ implies that

$$(Q_L + Q_R)S_m = (\lambda_L\gamma_m)_\mu T_L^\mu + (\lambda_R\gamma_m)^\mu T_{R\mu} \tag{7.14}$$

where T_L^μ and $T_{R\mu}$ are some ghost number three spinor states which are defined up to

$$\delta T_L^\mu = a\lambda_L^\mu + b_m(\gamma^m\lambda_R)^\mu + c^{np}(\gamma_{np}\lambda_L)^\mu, \quad \delta T_{R\mu} = a\lambda_{R\mu} + b_m(\gamma^m\lambda_L)_\mu + c^{np}(\gamma_{np}\lambda_R)_\mu\,. \tag{7.15}$$

After double dimensional reduction,

$$U^*_{\text{membrane}} = \lambda_L^\mu T_{R\mu} + \lambda_{R\mu}T_L^\mu\,, \tag{7.16}$$

which is invariant under (7.15) up to terms proportional to $\lambda\Gamma^c\lambda$ for $c = 1$ to 11. Furthermore, $(Q_L + Q_R)^2S_m = 0$ implies that $QU^*_{\text{membrane}} = 0$ up to terms proportional to $\lambda\Gamma^c\lambda$.

So, in this way, the unintegrated supermembrane and superstring vertex operators can be related to each other. For studying the M-theory conjecture for supermembrane scattering amplitudes, it would be useful to find a similar relation between the integrated supermembrane and superstring vertex operators.

Although only massless vertex operators have been discussed in this paper, it should be possible to use the cohomology of the supermembrane BRST operator to also define physical massive vertex operators. Unlike the superstring where the mass spectrum is discrete, one expects that physical vertex operators in the supermembrane BRST cohomology will have a continuous mass spectrum. This is because, to agree with results obtained using light-cone techniques, one expects these massive vertex operators to represent multiparticle states which can have a continuous spectrum. It would very interesting to verify this, perhaps using methods similar to those of [29] for the superstring.

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A. Zero momentum cohomology of $d = 11$ superparticle

In this appendix, the zero momentum cohomology of the $d = 11$ superparticle BRST operator, $Q = \lambda^\alpha d_\alpha$, will be computed for arbitrary ghost number and shown to correspond to the linearized $d = 11$ supergravity ghosts, fields, antifields and antighosts. As in the case of the $N = 1$ $d = 10$ superparticle discussed in the appendix of [13], the zero momentum cohomology of Q is equivalent to the "linear" cohomology of a nilpotent operator $\widetilde{Q}$ involving an unconstrained bosonic spinor $\widetilde{\lambda}^\alpha$ where

$$\widetilde{Q} = \widetilde{\lambda}^\alpha p_\alpha + \widetilde{\lambda}\Gamma^c\widetilde{\lambda} b_{(-1)c} + c^c_{(1)}(\widetilde{\lambda}\Gamma_{cd}\widetilde{\lambda} u^d_{(-2)} + \widetilde{\lambda}\Gamma^d\widetilde{\lambda} u_{(-2)[cd]}) + \cdots . \tag{A.1}$$

The new ghost variables $[b_{(-n)}, c_{(n)}]$ and $[u_{(-n)}, v_{(n)}]$ are fermionic and bosonic pairs of conjugate variables carrying ghost number $[-n, n]$ which substitute the pure spinor constraint on the λ^α variable, and "linear" cohomology signifies states in the cohomology of $\widetilde{Q}$ which are at most linearly dependent on these new variables.

Although $\widetilde{Q}$ for the $d = 11$ superparticle involves more terms than for the $N = 1$ $d = 10$ superparticle, the proof of equivalence of its "linear" cohomology with the zero momentum cohomology of $Q = \lambda^\alpha d_\alpha$ is identical to the proof in the appendix of [13] and will not be repeated here. As in the $N = 1$ $d = 10$ superparticle case described in (2.22), the term $\widetilde{\lambda}\Gamma^c\widetilde{\lambda} b_{(-1)c}$ in (A.1) imposes the pure spinor constraint and the remaining terms in (A.1) remove the extra gauge invariances implied by this constraint. As will now be shown for the $d = 11$ superparticle, these remaining terms involve new ghost variables with ghost numbers up to $[-7, 7]$.

The complete expression for $\widetilde{Q}$ for the $d = 11$ superparticle is

$$\begin{aligned}
\widetilde{Q} = {}& \widetilde{\lambda}^\alpha p_\alpha + \widetilde{\lambda}\Gamma^c\widetilde{\lambda} b_{(-1)c} + c^c_{(1)}\left(\widetilde{\lambda}\Gamma_{cd}\widetilde{\lambda} u^d_{(-2)} + \widetilde{\lambda}\Gamma^d\widetilde{\lambda} u_{(-2)[cd]}\right) + \\
& + v^c_{(2)}\left((\widetilde{\lambda}\Gamma_c)_\alpha b^\alpha_{(-2)} + \widetilde{\lambda}\Gamma^d\widetilde{\lambda} b_{(-3)(cd)}\right) + \\
& + v^{[cd]}_{(2)}\left(\frac{1}{2}(\widetilde{\lambda}\Gamma_{cd})_\alpha b^\alpha_{(-2)} + \eta_{de}\widetilde{\lambda}\Gamma^{ef}\widetilde{\lambda} b_{(-3)(cf)} + \widetilde{\lambda}\Gamma^e\widetilde{\lambda} b_{(-3)[cde]}\right) + \\
& + c^\alpha_{(2)}\left(-\widetilde{\lambda}\Gamma^c\widetilde{\lambda} u_{(-3)c\alpha} + \frac{1}{2}(\widetilde{\lambda}\Gamma^{cd})_\alpha(\widetilde{\lambda}\Gamma_c)^\beta u_{(-3)d\beta}\right) + \\
& + \frac{1}{2}c^{(de)}_{(3)}(\widetilde{\lambda}\Gamma_d)^\alpha u_{(-3)e\alpha} + \frac{1}{4}c^{[def]}_{(3)}(\widetilde{\lambda}\Gamma_{ef})^\alpha u_{(-3)d\alpha} + \\
& + v^{c\alpha}_{(3)} b^{d\beta}_{(-4)} M_{c\alpha\ d\beta\ \gamma\delta}\widetilde{\lambda}^\gamma\widetilde{\lambda}^\delta + \\
& + \frac{1}{4}u^{[def]}_{(-4)}(\widetilde{\lambda}\Gamma_{ef})^\alpha c_{(4)d\alpha} + \frac{1}{2}u^{(de)}_{(-4)}(\widetilde{\lambda}\Gamma_d)^\alpha c_{(4)e\alpha} + \\
& + u^\alpha_{(-5)}\left(-\widetilde{\lambda}\Gamma^c\widetilde{\lambda} c_{(4)c\alpha} + \frac{1}{2}(\widetilde{\lambda}\Gamma^{cd})_\alpha(\widetilde{\lambda}\Gamma_c)^\beta c_{(4)d\beta}\right) + \\
& + b^{[cd]}_{(-5)}\left(\frac{1}{2}(\widetilde{\lambda}\Gamma_{cd})_\alpha v^\alpha_{(5)} + \eta_{de}\widetilde{\lambda}\Gamma^{ef}\widetilde{\lambda} v_{(4)(cf)} + \widetilde{\lambda}\Gamma^e\widetilde{\lambda} v_{(4)[cde]}\right) + \\
& + b^c_{(-5)}\left((\widetilde{\lambda}\Gamma_c)_\alpha v^\alpha_{(5)} + \widetilde{\lambda}\Gamma^d\widetilde{\lambda} v_{(4)(cd)}\right) + \\
& + u^c_{(-6)}\left(\widetilde{\lambda}\Gamma_{cd}\widetilde{\lambda} c^d_{(5)} + \widetilde{\lambda}\Gamma^d\widetilde{\lambda} c_{(5)[cd]}\right) + b_{(-7)}\widetilde{\lambda}\Gamma^c\widetilde{\lambda} v_{(6)c} ,
\end{aligned} \tag{A.2}$$

where $M_{c\alpha\ d\beta\ \gamma\delta}$ are fixed coefficients which will be defined below. Before explaining the origin of the various terms in (A.2), it will be first be checked that the linear cohomology of $\widetilde{Q}$ corresponds to the zero momentum $d = 11$ supergravity ghosts, fields, antifields and antighosts.

Since $\widetilde{\lambda}^\alpha$ is unconstrained, the term $\widetilde{\lambda}^\alpha p_\alpha$ in (A.2) implies using the standard quartet mechanism that states in the cohomology of $\widetilde{Q}$ are independent of x^c and θ^α. So states in the "linear" cohomology are represented by the elements

$$\Big[1, c^c_{(1)}, v^c_{(2)}, v^{[cd]}_{(2)}, c^\alpha_{(2)}, c^{(cd)}_{(3)}, c^{[cde]}_{(3)}, v^{c\alpha}_{(3)}, c^{c\alpha}_{(4)}, v^{[cde]}_{(4)}, v^{(cd)}_{(4)}, v^\alpha_{(5)}, c^{[cd]}_{(5)}, c^c_{(5)}, v^c_{(6)}, c_{(7)}\Big], \tag{A.3}$$

which were shown in subsection 4.3 to correspond to the $d = 11$ supergravity ghosts, fields, antifields and antighosts. To map these elements to states in the zero momentum cohomology of Q, one needs to find BRST-closed expressions involving these elements which commute with $\widetilde{Q}$. For example, the BRST-closed expression corresponding to $c^c_{(1)}$ is $c^c_{(1)} - \widetilde{\lambda}\Gamma^c\theta$, the BRST-closed expression corresponding to $v^{[bc]}_{(2)}$ is $v^{[bc]}_{(2)} - c^{[b}_{(1)}\widetilde{\lambda}\Gamma^{c]}\theta + \frac{1}{2}(\widetilde{\lambda}\Gamma^b\theta)(\widetilde{\lambda}\Gamma^c\theta)$, and the BRST-closed expression corresponding to $v^c_{(2)}$ is $v^c_{(2)} - c_{(1)d}\widetilde{\lambda}\Gamma^{cd}\theta - \frac{1}{2}(\widetilde{\lambda}\Gamma^{cd}\theta)(\widetilde{\lambda}\Gamma_d\theta)$. The corresponding states in the zero momentum cohomology of Q are obtained by setting all new variables to zero and replacing $\widetilde{\lambda}^\alpha$ with λ^α, i.e. the state corresponding to $c^c_{(1)}$ is $-\lambda\Gamma^c\theta$, the state corresponding to $v^{[bc]}_{(2)}$ is $\frac{1}{2}(\lambda\Gamma^b\theta)(\lambda\Gamma^c\theta)$, and the state corresponding to $v^c_{(2)}$ is $-\frac{1}{2}(\lambda\Gamma^{cd}\theta)(\lambda\Gamma_d\theta)$.

Returning now to the explanation of the terms in $\widetilde{Q}$ of (A.2), the second term $\widetilde{\lambda}\Gamma^c\widetilde{\lambda}b_{(-1)c}$ enforces the pure spinor constraint and is invariant under $\delta b_{(-1)c} = \widetilde{\lambda}\Gamma^d\widetilde{\lambda}f_{[cd]} + \widetilde{\lambda}\Gamma_{cd}\widetilde{\lambda}g^d$ for arbitrary gauge parameters $f_{[cd]}$ and g^d. The third term in (A.2) fixes these gauge invariances, but introduces new gauge invariances of $u^d_{(-2)}$ and $u^{[cd]}_{(-2)}$ which are gauge-fixed by the fourth and fifth terms in (A.2). Note that only the symmetric part of $b_{(-3)(cd)}$ is needed in the fourth and fifth terms of (A.2) since the antisymmetric part $b_{(-3)[cd]}$ can be absorbed by a redefinition of $b^\alpha_{(-2)} \to b^\alpha_{(-2)} + (\widetilde{\lambda}\Gamma^{cd})^\alpha b_{(-3)[cd]}$.

At this point, the structure of $\widetilde{Q}$ is quite complicated, but one can use the correspondence between the new ghost variables and the $d = 11$ supergravity fields to help in the construction of the remaining terms in $\widetilde{Q}$. In fact, it will now be conjectured that all terms in $\widetilde{Q}$ can be deduced from the known linearized supersymmetry transformations of the $d = 11$ supergravity ghosts, fields, antifields and antighosts. Although this is not surprising since the BRST transformations generated by $\widetilde{Q}$ must be supersymmetric, a proof has not yet been constructed for the following conjecture. Nevertheless, it is straightforward to check that the conjecture is consistent with all terms that have been computed in $\widetilde{Q}$ using the gauge-fixing procedure.

Note that all terms in $\widetilde{Q}$ are either linear or quadratic in $\widetilde{\lambda}^\alpha$. The first conjecture is that terms linear in $\widetilde{\lambda}^\alpha$ describe the zero momentum linearized supersymmetry transformations of the $d = 11$ supergravity fields where $\widetilde{\lambda}^\alpha$ plays the role of the supersymmetry parameter. For example, the terms $\frac{1}{2}c^{(de)}_{(3)}(\widetilde{\lambda}\Gamma_d)^\alpha u_{(-3)e\alpha}$ and $\frac{1}{4}c^{[def]}_{(3)}(\widetilde{\lambda}\Gamma_{ef})^\alpha u_{(-3)d\alpha}$ describe the zero momentum linearized supersymmetry transformations of the $d = 11$ supergravity fields

$$\delta^\alpha g_{(de)} = \frac{1}{2}\Gamma^{\alpha\beta}_{(d}\chi_{e)\beta}\,, \qquad \delta^\alpha b_{[def]} = \frac{1}{4}(\Gamma_{[ef})^{\alpha\beta}\chi_{d]\beta}\,, \tag{A.4}$$

where $c^{(de)}_{(3)}$ is identified with the graviton $g^{(de)}$, $c^{[def]}_{(3)}$ is identified with the three-form $b^{[def]}$, and $u^\alpha_{(-3)d}$ is identified with the gravitino χ^α_d.

The second conjecture is that the terms in $\widetilde{Q}$ which are quadratic in $\widetilde{\lambda}^\alpha$ can be deduced from the anticommutator of two linearized supersymmetry transformations where $\widetilde{\lambda}^\alpha\widetilde{\lambda}^\beta$ plays the role of the supersymmetry parameters in the anticommutator. If the $d = 11$ supersymmetry algebra were closed off-shell, the anticommutator of two supersymmetry transformations acting on any supergravity field would be proportional to a translation, i.e. $\{\delta_\alpha, \delta_\beta\}\phi_I = \Gamma^c_{\alpha\beta}\partial_c\phi_I$ for any ϕ_I. However, since the supersymmetry algebra is only closed on-shell, the anticommutator of two supersymmetry transformations acting on a supergravity field can contain a term proportional to equations of motion, i.e. $\{\delta_\alpha, \delta_\beta\}\phi_I = \Gamma^c_{\alpha\beta}\partial_c\phi_I + M_{IJ\ \alpha\beta}\frac{\partial\mathcal{S}}{\partial\phi_J}$.

For $d = 11$ supergravity fields, $M_{IJ\ \gamma\delta}$ is non-vanishing when I and J correspond to gravitino fields, i.e.

$$\{\delta_\gamma, \delta_\delta\}\chi_{c\alpha} = \Gamma^d_{\gamma\delta}\partial_d\chi_{c\alpha} + M_{c\alpha\ d\beta\ \gamma\delta}\frac{\partial\mathcal{S}}{\partial\chi_{d\beta}} \tag{A.5}$$

where the coefficients $M_{c\alpha\ d\beta\ \gamma\delta}$ can be explicitly computed using the linearized supersymmetry transformations of the standard $d = 11$ supergravity action. From the second conjecture, this implies the term

$$v^{c\alpha}_{(3)}b^{d\beta}_{(-4)}M_{c\alpha\ d\beta\ \gamma\delta}\widetilde{\lambda}^\gamma\widetilde{\lambda}^\delta$$

in (A.2) where $v^{c\alpha}_{(3)}$ corresponds to the gravitino $\chi^{c\alpha}$ and $b^{d\beta}_{(-4)}$ corresponds to the gravitino antifield $\chi^*_{d\beta}$ whose BRST transformation is the linearized equation of motion $\partial\mathcal{S}/\partial\chi^{d\beta}$.

To give another example of the second conjecture, the term

$$c^\alpha_{(2)}\left(-\widetilde{\lambda}\Gamma^c\widetilde{\lambda}u_{(-3)c\alpha} + \frac{1}{2}(\widetilde{\lambda}\Gamma_{cd})_\alpha(\widetilde{\lambda}\Gamma^c)^\delta u^d_{(-3)\delta}\right) \tag{A.6}$$

in (A.2) can be deduced from the anticommutator of two supersymmetry transformations acting on the supersymmetry ghost ξ_α. Using $\delta^\alpha\rho_c = (\Gamma_c\xi)^\alpha$ and $\delta^\beta\xi_\alpha = -\frac{1}{2}\partial_b\rho_c(\Gamma^{bc})_\alpha^{\ \beta}$ where ρ_c is the reparameterization ghost, one finds that

$$\{\delta_\beta, \delta_\gamma\}\xi_\alpha = \Gamma^c_{\beta\gamma}\partial_c\xi_\alpha + \left(-\Gamma^c_{\beta\gamma}\partial_c\xi_\alpha + \frac{1}{2}(\Gamma_{cd})_{\alpha(\beta}\Gamma^{c\delta}_{\ \ \gamma)}\partial^d\xi_\delta\right). \tag{A.7}$$

So the term (A.6) in $\widetilde{Q}$ can be deduced from (A.7) where $c_{(2)\alpha}$ corresponds to the supersymmetry ghost ξ_α and $u^d_{(-3)\delta}$ corresponds to the gravitino χ^d_δ whose BRST variation is $\partial^d\xi_\delta$.

So one can use these two conjectures to deduce all terms in $\widetilde{Q}$ of (A.2), and one can explicitly check that this construction is consistent with the required gauge-fixing properties of the term in $\widetilde{Q}$. Furthermore, one can check that these conjectures are also consistent with the BRST operator of (2.22) for $d = 10$ super-Maxwell theory. Note that the terms in the second half of (A.2) are related to terms in the first half of (A.2) by exchanging fields with antifields and ghosts with antighosts, i.e. by exchanging $[b_{(-n)}, c_{(n)}]$ with $[v_{(7-n)}, u_{(n-7)}]$. The term $v^{c\alpha}_{(3)}b^{d\beta}_{(-4)}M_{c\alpha\ d\beta\ \gamma\delta}\widetilde{\lambda}^\gamma\widetilde{\lambda}^\delta$ is invariant under this exchange since $M_{c\alpha\ d\beta\ \gamma\delta} = M_{d\beta\ c\alpha\ \gamma\delta}$ in order that $\{\delta_\gamma, \delta_\delta\}\mathcal{S} = \frac{\partial\mathcal{S}}{\partial\chi^{c\alpha}}M_{c\alpha\ d\beta\ \gamma\delta}\frac{\partial\mathcal{S}}{\partial\chi^{d\beta}} = 0$ where $\mathcal{S}$ is the linearized $d = 11$ supergravity action.

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